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Stochastic Partial Differential Equations driven by Poisson type noise

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Contents

Introduction

In this diploma thesis we are going to show existence and uniqueness of solutions to stochastic partial differential equations on a Gelfand triple $V \subset H \subset V^*$ driven by Poisson type noise with locally monotone coefficients of the form

$$\begin{aligned} X(t) = & \ A(s, X(s)) \, \mathrm{d}s + B(s, X(s)) \, \mathrm{d}W(s) \\ & + \int_Z f(s, X(s-), z) \, \bar{\mu}(\mathrm{d}z, \mathrm{d}s), \\ X(0) = & \ X_0, \end{aligned} \tag{1}$$

on a finite time horizon, where W is a cylindrical Wiener process and $\bar{\mu}$ is a compensated Poisson random measure.

Stochastic partial differential equations with jump type noise, such as Lévy type pertubations or Poisson type noise, play an important role for modelling real world problems. In physical research, risk modelling and option pricing in finance, genetics and even climate based research these equations emerge to a greater extent, since a Lévy pertubation or the additional Poisson jump term offer more suitable modelling abilities than stochastic partial differential equations driven by a Wiener process solely, cf. [App04], [NØP09], [HIP09], [NR13] or [XFLZ13].
Therefore, it should come to no surprise that in the last few years there has been a high research interest in mathematics concerning existence and uniqueness of solutions to stochastic partial differential equations (abbr.: SPDE) driven by discontinuous jump terms, especially driven by Lévy type noise. For example one can have a look in [LR04], [Kno05], [PZ07], [RZ07], [App09], [NØP09], [MPR10], [Pr0] or [BLZ11] and the references therein to get a rough overview. This research with respect to jump type noise extends even to stochastic partial differential equations on separable Banach spaces, cf. [RZ06], and has been recently carried out for multi-valued maps, see [Ste12] and [LS14]. Earlier, SPDEs driven by general discontinuous martingales had been studied by Gyöngy and Krylov already, see [GK80], [GK82] and [Gy2].
Typical examples to equations of type-(1) are stochastic Burgers equations and the stochastic p-Laplace equations.

The result presented in Theorem 2.2.1 is based upon the paper of Brzeźniak, Liu and Zhu [BLZ11] and states, that under certain conditions, such as local monontonicity and coercivity, equation (1) has a unique (strong) solution in the sense of Defintion 2.1.1. Due to the Lévy–Itô decomposition, the class of SPDEs driven by Lévy type noise can be reduced to the class of SPDEs, where the stochastic pertubation term is a sum of a Wiener process and a compensated Poisson random measure as in (1), cf. Section D.1 in [Ste12] or Section 9 in [NØP09].

More precisely, a stochastic partial differential equation of the type

$$X(t) = A(s, X(s))\,\mathrm{d}s + \sigma(s, X(s))\,\mathrm{d}L(s),$$

where L is a general Lévy process, can be written in the form of (1). However, contrary to the result in [BLZ11], we do not involve big jumps in our equation, which would cause the appearance of and additional summand in (1) driven by a general Poisson random measure.

The variational framework was commenly used (see e.g. [PR07]) to show existence and uniqueness to SPDEs driven by a cylindrical Wiener process, i.e. $f \equiv 0$ in (1), under the assumption that A and B are monotone operators and that A and B fulfill a coercivity condition. In [LR10] and [LR14] this result was improved by assuming that the operators A and B are only locally monotone. In [BLZ11] this approach led to existence and uniqueness for (1) under the further assumption, that also f is locally monotone. However, there is no need for f to fulfill a coercivity condition, too. A further, recently published generalization is the use of a generalized coercivity condition on A and B, cf. [LR13], in case $f \equiv 0$ to handle the tamed 3D-Navier-Stokes equation.
Since this thesis is based upon [BLZ11], we do not cope with a generalized coercivity condition here, but use some minor but important changes – which are inspired from [LR14] – to improve the assumptions made in [BLZ11] and [LR10]. It is important to make note of our division of uniqueness and existence of solutions to (1) in Theorem 2.2.1 depending on the given assumptions, because the claimed uniqueness (and even existence) result in [BLZ11] does not follow directly in general from the assumptions made therein, cf. Remark 2.2.2.
Hence this work can be understood as an extension to [PR07] and [LR10] with respect to the compensated Poisson random measure-term and covers all the results therein. Moreover, one should not lose track of the fact that in case $B \equiv 0$ this thesis provides a tool to handle SPDEs of pure jump type (sometimes called pure Lévy jump type) as well.

The intention of this work is to prove existence and uniqueness of solutions to (1) under corrected and weakened assumptions in a comprehensible way and in all details. For a discussion on the assumptions, we refer to Remark 2.2.2. Furthermore this work will provide some applications proved in all details, too.

Although this thesis is meant to be self-contained, the reader is required to have knowledge of stochastic integration in Hilbert spaces as well as knowledge of (cylindrical) Wiener processes. In this thesis we mainly stick to the notation of Section 2 and 3 in [PR07]. Sobolev embeddings are used frequently in Chapter 3, but summarized in Appendix F. Nevertheless, it is recommended to know about functional analysis and weak convergence. Let us mention [Alt06] and [Bre10] as references in this area.

Let us briefly outline the structure of this thesis.

In Chapter 1 we will introduce Poisson point processes and Poisson random measures. Briefly we will recall all necessary fundamentals of the stochastic integration with respect to Poisson point processes.

Chapter 2 contains the main result of this thesis and its proof. After introducing the variational framework, we define what is meant by a solution to (1) and postulate the main assumptions (cf. conditions (A1)–(A4)). They lead to Theorem 2.2.1 and Remark 2.2.2 in which we discuss the differences between our assumptions and the familiar results in [BLZ11] and others. Afterwards we will give a short outline of the proof and finally prove existence of solutions to (1) and uniqueness.

In the last chapter of this thesis, Chapter 3, Theorem 2.2.1 is applied to semilinear and quasi-linear stochastic equations driven by Poisson type noise. In the first case one can think of a stochastic Burgers type equations. The second case will be the p-Laplace equation. Following the intention of this work, the verification of all assumptions made in Chapter 2 will be done in all details.

In section A–F of the appendix we will present all auxiliary results needed in Chapters 2 and 3 for completeness, in particular those, that are missing or claimed, but not proved in [BLZ11].

I would like to thank my supervisor Prof. Dr. Michael Röckner for leading me to the field of stochastic partial differential equations and his constant support in the past years. Special thanks are given to Dr. Simon Michel for his helpful comments. Finally, I am very grateful for the support of my family and my better and worse half, Jule.

1. Stochastic Integration with respect to Poisson Processes

In this chapter we will introduce the Poisson random measure and the Poisson point process. Afterwards we will establish the fundaments on stochastic integration with respect to a Poisson point process. Our main references are [Kno05] and [IW81].

1.1. Poisson Random Measures and Poisson Point Processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let $(S, \mathcal{S})$ be a measurable space and let $\mathbb{M}_{\bar{\mathbb{N}}}(S)$ denote the set of $\bar{\mathbb{N}} = \mathbb{N}_0 \cup \{+\infty\}$-valued measures on $(S, \mathcal{S})$. We write $\mathcal{B}(\mathbb{M}_{\bar{\mathbb{N}}}(S))$ to denote the smallest σ-field on $\mathbb{M}_{\bar{\mathbb{N}}}(S)$ such that all mappings $j_B \colon \mathbb{M}_{\bar{\mathbb{N}}}(S) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$ for $B \in \mathcal{S}$ are measurable, i.e.

$$\mathcal{B}(\mathbb{M}_{\bar{\mathbb{N}}}(S)) = \sigma(\mathbb{M}_{\bar{\mathbb{N}}}(S) \ni \mu \mapsto \mu(B) \mid B \in \mathcal{S}).$$

1.1.1 Definition. A map $\mu \colon \Omega \times \mathcal{S} \to \bar{\mathbb{N}}$ is called $\bar{\mathbb{N}}$-*valued random measure*, if

(i) $\mu(\omega, \cdot) \in \mathbb{M}_{\bar{\mathbb{N}}}(S)$ for each $\omega \in \Omega$ and

(ii) $\mu(\cdot, B)$ is an $\bar{\mathbb{N}}$-valued random variable on $(\Omega, \mathcal{F}, P)$ for all $B \in \mathcal{S}$.

For simplicity of notation we will write $\mu(B)$ instead of $\mu(\cdot, B)$.

1.1.2 Definition. An $\bar{\mathbb{N}}$-valued random measure μ is called *Poisson random measure* if the following conditions hold.

(i) For all $B \in \mathcal{S}$ with $\mathbb{E}[\mu(B)] < \infty$, $\mu(B) : \Omega \to \bar{\mathbb{N}}$ is a Poisson distributed random variable with parameter $\mathbb{E}[\mu(B)]$, i.e.

$$P(\mu(B) = n) = e^{-\mathbb{E}[\mu(B)]} \cdot \frac{(\mathbb{E}[\mu(B)])^n}{n!}$$

for all $n \in \bar{\mathbb{N}}$. If $\mathbb{E}[\mu(B)] = \infty$, then $\mu(B) = \infty$ P-a.s.

(ii) For any pairwise disjoint $B_1, \dots, B_n \in \mathcal{S}$, $n \in \mathbb{N}$, the random variables $\mu(B_1), \dots, \mu(B_n)$ are independent.

Now let $(Z, \mathcal{Z})$ be another measurable space.

1.1.3 Definition. A *point function* $\mathfrak{p}$ on Z is a mapping $\mathfrak{p} \colon \mathcal{D}_{\mathfrak{p}} \subset (0, \infty) \to Z$, where the domain $\mathcal{D}_{\mathfrak{p}}$ of $\mathfrak{p}$ is countable.

1.1.4 Remark. *Each point function* $\mathfrak{p}$ *on* Z *induces a measure* $\mu_{\mathfrak{p}}(\mathrm{d}t, \mathrm{d}z)$ *on* $\big((0,\infty) \times Z, \mathcal{B}((0,\infty)) \otimes \mathcal{Z}\big)$. *Let* $\mathcal{P}(\mathcal{D}_{\mathfrak{p}})$ *denote the power set of* $\mathcal{D}_{\mathfrak{p}}$ *and let* $\tilde{\mathfrak{p}} \colon \mathcal{D}_{\mathfrak{p}} \to (0,\infty) \times Z,\ t \mapsto (t, \mathfrak{p}(t))$. *Let* ν *be the counting measure on* $(\mathcal{D}_{\mathfrak{p}}, \mathcal{P}(\mathcal{D}_{\mathfrak{p}}))$ *defined by* $\nu(A) = \#A$ *for all* $A \in \mathcal{P}(\mathcal{D}_{\mathfrak{p}})$. *Now let us define the measure*

$$\mu_{\mathfrak{p}}(A \times B) := \nu\left(\tilde{\mathfrak{p}}^{-1}(A \times B)\right)$$

for all $A \in \mathcal{B}((0,\infty))$ *and* $B \in \mathcal{Z}$. *Then we have*

$$\mu_{\mathfrak{p}}(A \times B) = \#\{t \in \mathcal{D}_{\mathfrak{p}} \,|\, t \in A,\ \mathfrak{p}(t) \in B\}.$$

Notation. *For given* $t \in (0,\infty)$ *and* $B \in \mathcal{Z}$ *we will set* $A =]0,t]$ *and write*

$$\mu_{\mathfrak{p}}(t, B) = \mu_{\mathfrak{p}}(]0,t] \times B).$$

Let

$$\mathcal{P}_Z = \{\mathfrak{p} \colon \mathcal{D}_{\mathfrak{p}} \subset (0,\infty) \to Z \,|\, \mathcal{D}_{\mathfrak{p}} \text{ is countable}\}$$

be the space of all point functions on Z and define

$$\mathcal{B}_{\mathcal{P}_Z} := \sigma\left(\mathcal{P}_Z \ni \mathfrak{p} \mapsto \mu_{\mathfrak{p}}(t, B) \,|\, t > 0,\ B \in \mathcal{Z}\right).$$

1.1.5 Definition. (i) A random variable $\mathfrak{p} \colon (\Omega, \mathcal{F}) \to (\mathcal{P}_Z, \mathcal{B}_{\mathcal{P}_Z})$ is called *point process on* Z *and* $(\Omega, \mathcal{F}, P)$.

(ii) Let θ_t be the shift operator given by $\theta_t \colon (0,\infty) \to (0,\infty)$, $s \mapsto s + t$. A point process $\mathfrak{p}$ is called *stationary* if for every $t > 0$ the process $\mathfrak{p}$ and the shifted process $\theta_t \mathfrak{p}$ have the same probability laws.

(iii) A point process $\mathfrak{p}$ is called *σ-finite* if there exists a sequence $(B_n)_{n \in \mathbb{N}} \subset \mathcal{Z}$ with $B_n \nearrow Z$ as $n \to \infty$ and

$$\mathbb{E}[\mu_{\mathfrak{p}}(t, B_n)] < \infty$$

for all $t > 0$ and $n \in \mathbb{N}$.

(iv) A *Poisson point process* is a point process $\mathfrak{p}$ on Z if there exists a Poisson random measure ν on $((0,\infty) \times Z, \mathcal{B}((0,\infty)) \otimes \mathcal{Z})$ and a P-zero set $N \in \mathcal{F}$ such that for all $\omega \in N^{\complement}$ and all $A \in \mathcal{B}((0,\infty))$, $B \in \mathcal{Z}$

$$\mu_{\mathfrak{p}(\omega)}(A \times B) = \nu(\omega, A \times B).$$

1.1.6 Proposition. *Let* $\mathfrak{p}$ *be a* σ*-finite Poisson point process on* Z *and* $(\Omega, \mathcal{F}, P)$. *Then* $\mathfrak{p}$ *is stationary if and only if there exists a* σ*-finite measure* m *on* $(Z, \mathcal{Z})$ *such that*

$$\mathbb{E}[\mu_{\mathfrak{p}}(\mathrm{d}t, \mathrm{d}z)] = \mathrm{d}t \otimes m(\mathrm{d}z).$$

Here, $\mathrm{d}t$ *denotes the Lebesgue-measure on* $((0,\infty), \mathcal{B}((0,\infty)))$. *In this case the measure* m *is uniquely determined and we call it the* characteristic measure of $\mu_{\mathfrak{p}}$.

Proof. See [Kno05, Proposition 2.10]. □

1.1.7 Definition. Let $\mathcal{F}_t$, $t \geqslant 0$ be a filtration on $(\Omega, \mathcal{F}, P)$ and $\mathfrak{p}$ a point process on Z and $(\Omega, \mathcal{F}, P)$.

(i) The process $\mathfrak{p}$ is called $(\mathcal{F}_t)$-*adapted*, if for every $t \geqslant 0$ and $B \in \mathcal{Z}$, $\mu_{\mathfrak{p}}(t, B)$ is $(\mathcal{F}_t)$-measurable.

(ii) The process $\mathfrak{p}$ is called an $(\mathcal{F}_t)$-*Poisson point process*, if it is $(\mathcal{F}_t)$-adapted and σ-finite, such that
$$\{\mu_{\mathfrak{p}}(]t, t+h] \times B) \,|\, h > 0,\ B \in \mathcal{Z}\}$$
is independent of $\mathcal{F}_t$ for all $t \geqslant 0$.

Further, we define the set
$$\Gamma_{\mu_{\mathfrak{p}}} := \{B \in \mathcal{Z} \,|\, \mathbb{E}[\mu_{\mathfrak{p}}(t, B)] < \infty \text{ for all } t > 0\}.$$

1.1.8 Definition. Let $\mathcal{F}_t$ be a right-continuous filtration on $(\Omega, \mathcal{F}, P)$. Let $\mathfrak{p}$ be a Poisson point process on Z and $(\Omega, \mathcal{F}, P)$. The process $\mathfrak{p}$ is said to be *of class (QL)* or *quasi-left-continuous with respect* $\mathcal{F}_t$, if it is $(\mathcal{F}_t)$-adapted and σ-finite and for all $B \in \mathcal{Z}$ there exists a process $\nu(t, B) : \Omega \to \mathbb{R}$, $t \geqslant 0$, such that the following conditions hold:

(1) If $B \in \Gamma_{\mu_{\mathfrak{p}}}$, the process $\nu(t, B)$, $t \geqslant 0$, is a continuous $(\mathcal{F}_t)$-adapted increasing process with $\nu(0, B) = 0$ P.-a.s.

(2) For all $t \geqslant 0$ and for P-a.e. $\omega \in \Omega$, $\nu(\omega)(t, \cdot)$ is a σ-finite measure on $(Z, \mathcal{Z})$.

(3) If $B \in \Gamma_{\mu_{\mathfrak{p}}}$, then
$$\bar{\mu}_{\mathfrak{p}}(t, B) := \mu_{\mathfrak{p}}(t, B) - \nu(t, B), \quad t \geqslant 0,$$
is an $(\mathcal{F}_t)$-martingale.

In this case we call ν the *compensator of* $\mu_{\mathfrak{p}}$ and $\bar{\mu}_{\mathfrak{p}}$ is called *compensated Poisson random measure of* $\mu_{\mathfrak{p}}$.

1.1.9 Proposition. *Let $\mathcal{F}_t$, $t \geqslant 0$, be a right-continuous filtration on $(\Omega, \mathcal{F}, P)$. Let m be a σ-finite measure on $(Z, \mathcal{Z})$ and let $\mathfrak{p}$ be a stationary $(\mathcal{F}_t)$-Poisson point process on Z with characteristic measure m.*
Then $\mathfrak{p}$ is quasi-left-continuous with respect to $\mathcal{F}_t$ and with compensator
$$\nu(t, B) = t \cdot m(B), \quad t \geqslant 0,\ B \in \mathcal{Z}.$$

1.2. Stochastic Integration with respect to a Poisson Point Process

In this section we want to construct the stochastic integral with respect to compensated Poisson random measures, where the random measure is induced by a stationary Poisson point process.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with normal filtration $(\mathcal{F}_t)$, $t \geqslant 0$, and let $(Z, \mathcal{Z})$ be another measurable space with a σ-finite measure m.
We fix a stationary $(\mathcal{F}_t)$-Poisson point process $\mathfrak{p}$ on Z as defined in the previous section with characteristic measure m. Since $\mathfrak{p}$ is stationary and by Proposition 1.1.9, $\mathfrak{p}$ is quasi-left-continuous with respect to $\mathcal{F}_t$ and the compensator ν of the induced measure $\mu_{\mathfrak{p}}$ is given by $\nu = \mathrm{d}t \otimes m$. The compensated Poisson random measure is given by

$$\bar{\mu}_{\mathfrak{p}} = \mu_{\mathfrak{p}} - \nu = \mu_{\mathfrak{p}} - \mathrm{d}t \otimes m.$$

We will denote these measures simply by μ and $\bar{\mu}$, since $\mathfrak{p}$ is fixed throughout the whole section.
Further let $(H, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space, $T \in (0, \infty)$ and set

$$\Gamma = \{B \in \mathcal{Z} \,|\, m(B) < \infty\}.$$

Let $\mathcal{M}_2^T(H)$ be the space of all càdlàg square integrable martingales in H with respect to $(\mathcal{F}_t)$.

1.2.1 Definition. An H-valued process $\Phi(t) : \Omega \times Z \to H$, $t \in [0, T]$, is an *elementary process*, if there exists a partition $0 = t_0 < t_1 < \cdots < t_k = T$, $k \in \mathbb{N}$ and for $m \in \{0, k-1\}$ there exist pairwise disjoint $B_1^m, \ldots, B_{n_m}^m \in \Gamma$, $n_m \in \mathbb{N}$, and functions $\Phi_i^m \in L^2(\Omega, \mathcal{F}_{t_m}, P; H)$, $0 \leqslant i \leqslant n_m$, such that the following holds:

$$\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{n_m} \Phi_i^m \mathbb{I}_{]t_m, t_{m+1}] \times B_i^m}.$$

The linear space of all elementary processes is denoted by $\mathcal{E}$.

The stochastic integral with respect to $\bar{\mu}$ can now be defined for an elementary process $\Phi \in \mathcal{E}$ and $t \in [0, T]$ by

$$\begin{aligned} \operatorname{Int}(\Phi)(t) &:= \int_{]0,t]} \int_Z \Phi(s, z)\, \bar{\mu}(\mathrm{d}t, \mathrm{d}z) \\ &:= \sum_{m=0}^{k-1} \sum_{i=1}^{n_m} \Phi_i^m \left(\bar{\mu}(t_{m+1} \wedge t, B_i^m) - \bar{\mu}(t_m \wedge t, B_i^m) \right). \end{aligned}$$

Then $\operatorname{Int}(\Phi)$ is linear in $\Phi \in \mathcal{E}$ and P-a.s. well defined. We set

$$\|\Phi\|_T^2 := \mathbb{E}\left[\int_{]0,t]} \int_Z \|\Phi(s, z)\|_H^2 \, m(\mathrm{d}z)\, \mathrm{d}s \right]$$

for $\Phi \in \mathcal{E}$.

1.2.2 Proposition. *For $\Phi \in \mathcal{E}$ we have $\operatorname{Int}(\Phi) \in \mathcal{M}_T^2(H)$, $\operatorname{Int}(\Phi)(0) = 0$ P-a.s. For all $t \in [0, T]$*

$$\mathbb{E}\left[\|\operatorname{Int}(\Phi)(t)\|_H^2 \right] = \mathbb{E}\left[\int_{]0,t]} \int_Z \|\Phi(s, z)\|_H^2 \, m(\mathrm{d}z)\, \mathrm{d}s \right] \tag{1.2.1}$$

holds. In other words, $\mathrm{Int}\colon \left(\mathcal{E}, \|\cdot\|_T^2\right) \to \left(\mathcal{M}_T^2(H), \|\cdot\|_{\mathcal{M}_T^2}\right)$ *is an isometry with*

$$\|\mathrm{Int}(\Phi)\|_{\mathcal{M}_T^2} = \|\Phi\|_T^2 .$$

Proof. See [Kno05, Proposition 2.22]. □

Up to this point, $\|\cdot\|_T$ is only a seminorm on $\mathcal{E}$. Thus let us consider the space of equivalence classes of elementary processes with respect to $\|\cdot\|_T$ and let us denote it again by $\mathcal{E}$ for simplicity of notation. $\mathcal{E}$ is dense in the completion $\overline{\mathcal{E}}^{\|\cdot\|_T}$ of $\mathcal{E}$ with respect to $\|\cdot\|_T$ and hence there exists a unique isometric extension of Int to $\overline{\mathcal{E}}^{\|\cdot\|_T}$ and the isometry in (1.2.1) also holds for each process in $\overline{\mathcal{E}}^{\|\cdot\|_T}$.
The following proposition will characterize $\overline{\mathcal{E}}^{\|\cdot\|_T}$. But first we need to define the predictable σ-algebra on $[0,T]\times\Omega\times Z$ by

$$\begin{aligned}\mathcal{P}_T(Z) &:= \sigma\left(g\colon\ [0,T]\times\Omega\times Z \to \mathbb{R} \mid g \text{ is } (\mathcal{F}_t\otimes\mathcal{Z})\text{-adapted and left-continuous}\right)\\ &= \sigma\left(\{]s,t]\times F_s\times B \mid 0\leqslant s\leqslant t\leqslant T,\ F_s\in\mathcal{F}_s,\ B\in\mathcal{Z}\}\right.\\ &\qquad \left.\cup\,\{\{0\}\times F_0\times B \mid F_0\in\mathcal{F}_0,\ B\in\mathcal{Z}\}\right)\end{aligned}$$

and set

$$\begin{aligned}\mathcal{N}_{\bar\mu}^2(T,Z;H) := \Bigg\{\Phi\colon\ [0,T]\times\Omega\times Z\to H\ \Bigg|\ &\Phi \text{ is } \mathcal{P}_T(Z)/\mathcal{B}(H)\text{-measurable}\\ &\text{and } \|\Phi\|_T = \mathbb{E}\left[\int_{]0,T]}\int_Z \|\Phi(s,z)\|_H^2\ m(\mathrm{d}z)\ \mathrm{d}s\right]^{\frac{1}{2}} < \infty\Bigg\}.\end{aligned}$$

1.2.3 Proposition. *In the situation above we have*

$$\overline{\mathcal{E}}^{\|\cdot\|_T} = \mathcal{N}_{\bar\mu}^2(T,Z;H)$$

and

$$\mathcal{N}_{\bar\mu}^2(T,Z;H) = L^2\left([0,T]\times\Omega\times Z,\ \mathcal{P}_T(Z),\ \mathrm{d}t\otimes P\otimes m;\ H\right).$$

Proof. See [Kno05, Proposition 2.24]. □

1.2.1. Properties of the Poisson integral

We will now collect some important properties of the stochastic integral with respect to a compensated Poisson random measure.

1.2.4 Proposition. *Let* $\Phi\in\mathcal{N}_{\bar\mu}(T,Z;H)$ *and let* τ *be an* $(\mathcal{F}_t)$*-stopping time with* $P(\tau\leqslant T) = 1$*. Then* $\mathbb{I}_{]0,\tau]}\Phi\in\mathcal{N}_{\bar\mu}^2(T,Z;H)$ *and*

$$\int_{]0,t]}\int_Z \mathbb{I}_{]0,\tau]}(s)\,\Phi(s,z)\ \bar\mu(\mathrm{d}s,\mathrm{d}z) = \int_{]0,t\wedge\tau]}\int_Z \Phi(s,z)\ \bar\mu(\mathrm{d}s,\mathrm{d}z) \qquad P\text{-a.s.}$$

for all $t\in[0,T]$.

Proof. See [Kno05, Proposition 3.5]. □

1.2.5 Proposition. *Let* $\Phi \in \mathcal{N}_{\bar{\mu}}(T, Z; H)$ *and set*

$$X(t) := \int_{]0,t]} \int_Z \Phi(s,z)\, \bar{\mu}(\mathrm{d}s, \mathrm{d}z), \quad t \in [0,T].$$

Then X *is cádlág and* $X(t) = X(t-)$ P*-a.s. for all* $t \in [0,T]$.

Proof. See [Kno05, Proposition 3.6]. □

1.2.6 Proposition. *Let* $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z; H)$, $\tilde{H}$ *be another Hilbert space and let* $L \in L\left(H; \tilde{H}\right)$. *Then* $L(\Phi) \in \mathcal{N}^2_{\bar{\mu}}\left(T, Z; \tilde{H}\right)$ *and*

$$L\left(\int_{]0,t]} \int_Z \Phi(s,z)\, \bar{\mu}(\mathrm{d}s, \mathrm{d}z)\right) = \int_{]0,t]} \int_Z L(\Phi(s,z))\, \bar{\mu}(\mathrm{d}s, \mathrm{d}z) \quad P\text{-a.s.}$$

for all $t \in [0,T]$.

Proof. See [Kno05, Proposition 3.7]. □

1.2.7 Proposition. *Let* $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z; H)$. *Then for all* $t \in [0,T]$

$$\mathbb{E}\left[\int_{]0,t]} \int_Z \Phi(s,z)\, \mu(\mathrm{d}s, \mathrm{d}z)\right] = \mathbb{E}\left[\int_{]0,t]} \int_Z \Phi(s,z)\, m(\mathrm{d}z)\, \mathrm{d}s\right].$$

Proof. See [Ste12, Proposition 2.21]. □

Let $[X]_t$ denote the square bracket of an H-valued process $X(t)$.

1.2.8 Proposition. *Let* $\Phi \in \mathcal{N}^2_{\bar{\mu}}(T, Z; H)$ *and*

$$X(t) := \int_{]0,t]} \int_Z \Phi(s,z)\, \bar{\mu}(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0.$$

Then

$$[X]_t = \int_{]0,t]} \int_Z \|\Phi(s,z)\|^2_H\, \mu(\mathrm{d}s, \mathrm{d}z).$$

Proof. [Ste12, Corollary 2.23] □

2. Main Theorem

In this chapter we will formulate and prove the main theorem of this work. Our main reference is [BLZ11].

2.1. Setting and Assumptions

Let $(H, \langle \cdot, \cdot \rangle_H)$ be a separable real Hilbert space identified with its dual space H^* by the Riesz isomorphism. Let V be a real reflexive Banach space with dual space V^*, such that V is continuously embedded into H, i.e. there exists $\bar{C} > 0$ with

$$\|v\|_H \leqslant \bar{C}\|v\|_V \quad \text{for all } v \in V,$$

and such that V is dense in H. We call (V, H, V^*) a Gelfand triple. It follows that $H^* \subset V^*$ continuously and densly (cf. [Zei90, Proposition 23.13]) and also

$$V \subset H \equiv H^* \subset V^*$$

continuously and densly. If ${}_{V^*}\langle \cdot, \cdot \rangle_V$ denotes the duality between V and V^*, then we have

$${}_{V^*}\langle u, v \rangle_V = \langle u, v \rangle_H \quad \text{for all } u \in H,\ v \in V.$$

Note that V^* is separable since $H \subset V^*$ continuously and densly and hence this is true for V.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with normal filtration $(\mathcal{F}_t)$, $t \geqslant 0$. Let $(Z, \mathcal{Z}, m)$ be a measurable space with a σ-finite measure m. As in Section 1.2 we fix a stationary $(\mathcal{F}_t)$-Poisson point process $\mathfrak{p}$ on Z and $(\Omega, \mathcal{F}, P)$. The compensated Poisson random measure is given by

$$\bar{\mu}(t, B) = \mu(t, B) - tm(B), \quad t \geqslant 0,\ B \in \mathcal{Z},$$

where $\bar{\mu} = \bar{\mu}_{\mathfrak{p}}$ and $\mu = \mu_{\mathfrak{p}}$. Let U be another separable Hilbert space and let $(W_t)_{t\geqslant 0}$ be a U-valued cylindrical Wiener process on the probability space $(\Omega, \mathcal{F}_t, P)$. Let $0 < T < \infty$ be fixed.

We consider the stochastic partial differential equation of the following type

$$\begin{aligned} \mathrm{d}X(t) =& A(t, X(t))\,\mathrm{d}t + B(t, X(t))\,\mathrm{d}W(t) \\ &+ \int_Z f(t, X(t-), z)\,\bar{\mu}(\mathrm{d}t, \mathrm{d}z), \qquad (2.1.1) \\ X(0) =& X_0, \end{aligned}$$

where X_0 is an $\mathcal{F}_0$-measurable random variable. We consider the operators

$$\begin{aligned} A\colon\ & [0,T]\times\Omega\times V && \to V^*, \\ B\colon\ & [0,T]\times\Omega\times V && \to L_2\left(U,H\right), \\ f\colon\ & [0,T]\times\Omega\times V\times Z && \to H, \end{aligned}$$

where $\left(L_2\left(U;H\right),\|\cdot\|_{L_2}\right)$ denotes the space of Hilbert-Schmidt operators from U to H. For simplicity we write $A\left(t,v\right)$ for the mapping $\omega\mapsto A\left(t,\omega,v\right)$ and analogously for B and f. The operators A and B are both assumed to be progressively measurable, i.e. for all $t\in[0,T]$ these maps restricted to $[0,t]\times\Omega\times V$ are $\mathcal{B}\left([0,t]\right)\otimes\mathcal{F}_t\otimes\mathcal{B}\left(V\right)$-measurable where $\mathcal{B}$ denotes the Borel-σ-algebra. f is assumed to be a $\mathcal{P}\otimes\mathcal{B}\left(V\right)\otimes\mathcal{Z}$-measurable function, where $\mathcal{P}$ is the predictable σ-algebra which is generated by all left-continuous and $\mathcal{F}$-adapted real-valued processes on $[0,T]\times\Omega$.
We assume that there exist constants

$$\begin{aligned} \alpha>1, &\quad \beta\geqslant 0, \\ \theta>0, &\quad K>0, \end{aligned}$$

a non-negative, $\mathcal{F}$-adapted process $\left(F_t\right)_{t\in[0,T]}$ such that $F\in L^1\left([0,T]\times\Omega,\mathrm{d}t\otimes P;\mathbb{R}\right)$ and a measurable, hemicontinuous function $\varrho\colon V\to[0,\infty)$, which is locally bounded in V. Furthermore we assume that these constants and functions fulfill the following conditions for all v, v_1, $v_2\in V$, $\omega\in\Omega$ and all $t\in[0,T]$:

(A1) **Hemicontinuity**. The map

$$s\mapsto {}_{V^*}\langle A\left(t,v_1+sv_2\right),v\rangle_V$$

is continuous in $\mathbb{R}$.

(A2) **Local monotonicity**.

$$\begin{aligned} &2\,{}_{V^*}\langle A\left(t,v_1\right)-A\left(t,v_2\right),v_1-v_2\rangle_V+\|B\left(t,v_1\right)-B\left(t,v_2\right)\|_{L_2}^2 \\ &+\int_Z\|f\left(t,v_1,z\right)-f\left(t,v_2,z\right)\|_H^2\;m\left(\mathrm{d}z\right)\leqslant\left(F_t+\varrho\left(v_2\right)\right)\|v_1-v_2\|_H^2 . \end{aligned}$$

(A3) **Coercivity**.

$$2\,{}_{V^*}\langle A\left(t,v\right),v\rangle_V+\|B\left(t,v\right)\|_{L_2}^2+\theta\,\|v\|_V^\alpha\leqslant F_t+K\|v\|_H^2.$$

(A4) **Growth**.

$$\|A\left(t,v\right)\|_{V^*}^{\frac{\alpha}{\alpha-1}}\leqslant\left(F_t+K\|v\|_V^\alpha\right)\left(1+\|v\|_H^\beta\right).$$

2.1.1 Definition (Solution)**.** A *solution to (2.1.1)* is an $\mathcal{F}_t$-adapted, H-valued, càdlàg process $\left(X_t\right)_{t\in[0,T]}$, if for its $\mathrm{d}t\otimes P$-equivalent class $\bar{X}$ the following conditions hold:

(i) P-a.s. we have $\bar{X} \in L^{\alpha}([0,T] \times \Omega, \mathrm{d}t \otimes P; V) \cap L^2([0,T] \times \Omega, \mathrm{d}t \otimes P; H)$.

(ii) The following equality holds P-a.s. for all $t \in [0,T]$:

$$X(t) = X_0 + \int_0^t A\left(s, \bar{X}(s)\right) \mathrm{d}s + \int_0^t B\left(s, \bar{X}(s)\right) \mathrm{d}W(s) + \int_0^t \int_Z f\left(s, \bar{X}(s-), z\right) \bar{\mu}(\mathrm{d}s, \mathrm{d}z).$$

The integrability of all occuring integrals is required.

2.1.2 Remark. *Although A is an V^*-valued process by definition, we will see in Proposition 2.3.10 that the V^*-valued Bochner integral with respect to $\mathrm{d}t$ will become H-valued.*

Our main aim in this chapter will be to establish existence and uniqueness of strong solutions to (2.1.1) in the sense of Definition 2.1.1.

2.2. Formulation of the Theorem

Suppose all conditions and assumptions from Section 2.1 hold. Let $C_{BDG} > 0$ be the generic constant from the Burkholder-Davis-Gundy inequality D.5 (i) in case $p = 1$ and define

$$\begin{aligned}\Gamma &:= \Gamma(\theta, \beta, C_{BDG}) \\ &:= \theta \frac{\beta+2}{2}\left[(\beta+2)\left(\beta + \frac{3}{2}(\beta+2) C_{BDG}^2 + 2^{\beta+2} + 1\right) - 3 \cdot 2^{\beta+1}\right]^{-1}. \end{aligned} \tag{2.2.1}$$

Then $\Gamma > 0$ because $\theta, C_{BDG} > 0$, $\beta \geqslant 0$ and $(\beta+2)\,2^{\beta+2} = 2(\beta+2)\,2^{\beta+1} \geqslant 4 \cdot 2^{\beta+1} > 3 \cdot 2^{\beta+1}$. We can now formulate the main theorem of this work.

2.2.1 Theorem. *Suppose that conditions (A1) - (A4) are satisfied and that $F \in L^{\frac{\beta+2}{2}}([0,T] \times \Omega, \mathrm{d}t \times P)$. Suppose there exist constants $0 \leqslant \gamma < \theta \frac{\beta+2}{2} \cdot \left[(\beta+2)(\beta+1) + 2^{\beta+1}(2\beta+1)\right]^{-1}$ and $C > 0$ such that*

$$\|B(t,v)\|_{L_2}^2 + \int_Z \|f(t,v,z)\|_H^2 \, m(\mathrm{d}z) \leqslant C\left(1 + F_t + \|v\|_H^2\right) + \gamma \|v\|_V^{\alpha}, \tag{B1}$$

$$\int_Z \|f(t,v,z)\|_H^{\beta+2} \, m(\mathrm{d}z) \leqslant C\left(1 + F_t^{\frac{\beta+2}{2}} + \|v\|_H^{\beta+2}\right) + \gamma \|v\|_H^{\beta} \|v\|_V^{\alpha}, \tag{B2}$$

and let ϱ be such that

$$\varrho(v) \leqslant C\left(1 + \|v\|_V^{\alpha}\right)\left(1 + \|v\|_H^{\beta}\right) \tag{B3}$$

for every $0 \leqslant t \leqslant T$, $\omega \in \Omega$ and $v \in V$. Then equation (2.1.1) has a solution $(X_t)_{t \in [0,T]}$ for every initial value $X_0 \in L^{\bar{\beta}}(\Omega, \mathcal{F}_0, P; H)$, where $\bar{\beta} \geqslant \beta + 2$. Furthermore, *(i) there exists a constant* $\tilde{C} = \tilde{C}(p, \gamma, \theta, C, K, T) > 0$ *such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X(t)\|_H^{\beta+2}\right] \leqslant \tilde{C}\left(1 + \mathbb{E}\left[\|X_0\|_H^{\beta+2}\right] + \mathbb{E}\left[\int_0^T F_t^{\frac{\beta+2}{2}}\,\mathrm{d}t\right]\right).$$

(ii) if $0 \leqslant \gamma < \Gamma$*, then there exists a constant* $\hat{C} = \hat{C}(p, \gamma, \theta, C, C_{BDG}, K, T) > 0$ *such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|_H^{\beta+2}\right] \leqslant \hat{C}\left(1 + \mathbb{E}\left[\|X_0\|_H^{\beta+2}\right] + \mathbb{E}\left[\int_0^T F_t^{\frac{\beta+2}{2}}\,\mathrm{d}t\right]\right)$$

and the solution $X = (X_t)_{t\in[0,T]}$ *is unique.*

Let us do a short discussion on this theorem and its assumptions in contrast to the familiar results in [BLZ11, LR10].

2.2.2 Remark.

(i) Although stated otherwise in [BLZ11, Theorem 1.2], the claimed result for uniqueness of the solution $X = (X_t)_{t\in[0,T]}$ *in the case when 2.2.1 (ii) does not hold, i.e.* $\gamma \geqslant \Gamma$*,* cannot *be achieved. We will see in the proof of uniqueness in Section 2.4 that we need Corollary 2.3.13 (iii), which only holds for* $\gamma < \Gamma$*.*

(ii) Contrary to the statement in [BLZ11, Theorem 1.2], we do not *state in 2.2.1 (i) and (ii) that*

$$\mathbb{E}\left[\int_0^T \|X(t)\|_V^{\alpha} \|X(t)\|_H^{\beta}\,\mathrm{d}t\right]$$

is bounded. Although we will see that this term is bounded in case of the finite dimensional equation, typical convergence arguments are insufficient to show that the term above is also bounded in the infinite dimensional case.

(iii) The given bound on γ *in Theorem 2.2.1 is of a technical origin, as one will see in the proof of Lemma 2.3.5 (ii). One can use*

$$\gamma < \frac{\theta}{2}\left[(\beta+2) + 2^{\beta+2}\right]^{-1}$$

as a better looking, but smaller bound in Theorem 2.2.1, because

$$(\beta+2)(\beta+1) + 2^{\beta+1}(2\beta+1) \leqslant (\beta+2)^2 + 2^{\beta+1}(2\beta+4) = (\beta+2)^2 + (\beta+2)\,2^{\beta+2}$$

implies that

$$\begin{aligned}\gamma &< \frac{\theta}{2}\left[(\beta+2) + 2^{\beta+2}\right]^{-1} = \theta\frac{\beta+2}{2}\left[(\beta+2)^2 + (\beta+2)\,2^{\beta+2}\right]^{-1} \\ &\leqslant \theta\frac{\beta+2}{2}\cdot\left[(\beta+2)(\beta+1) + 2^{\beta+1}(2\beta+1)\right]^{-1}\end{aligned}$$

and so the Theorem holds.

(iv) *We only know that C_{BDG} is generic. Since no information about its calculation can be found in [Kal97], we cannot calculate Γ explicity here. But from [LS89] we know that $C_{BDG} \leqslant 3$, since we need it for $p = 1$.*

(v) *Condition (B2) is weaker than condition (1.3) in [BLZ11, Theorem 1.2], because we allow $\int_Z \|f(t,v,z)\|_H^{\beta+2} m(\mathrm{d}z)$ also to be bounded by $\|v\|_H^\beta \|v\|_V^\alpha$. As a consequence we cannot choose $\gamma < \infty$ to be arbitrary if $\beta = 0$, but we can still choose $\gamma < \theta$ if $\beta = 0$ (cf. Remark 2.3.6 (ii)) and our bound on γ becomes smaller.*

Anyway, the claimed bound of $\gamma < \frac{\theta}{2\beta}$ in [BLZ11, Theorem 1.2] is not sufficient for $\beta \in \left(0, \frac{1}{2}\right)$ to show existence with the methods used therein. The reason lies in the missing analogue of our Lemma 2.3.5 (i) in [BLZ11], which has not been worked out there. We can see in the proof that we always need $\gamma < \theta$ to apply Gronwall's inequality, which is e.g. obviously not true for $\beta = \frac{1}{4}$ (then we could choose $\theta < \gamma < 2\theta$).

Another reason for weakening condition (B2) here and therefore losing a higher bound on γ is the fact, that all the examples in [BLZ11] do not even involve γ, i.e. $\gamma = 0$ there. In Chapter 3, we are able to use this bound to claim more general conditions.

(vi) *We use a weaker local monotonicity condition (A2) here. For condition (H2) in [BLZ11] or [LR10] the bound is given by*

$$(K + \varrho(v_2)) \|v_1 - v_2\|_H^2$$

in (A2). Here, we allow $(F_t)_{t\in[0,T]}$ to be part of the bound instead. This generalization is inspired from [LR14], which is not published yet.

(vii) *In a special case, namely for $\beta = 0$, $\alpha < \left[1 - \frac{5\theta}{16K}\right]^{-1}$ and $5\theta < 16K$, we can deduce (B1) from (A3) and (A4). Indeed, both (A3) and (A4) imply*

$$\|B(t,v)\|_{L_2}^2 \leqslant -\theta \|v\|_V^\alpha + F_t + K\|v\|_H^2 + 2\left((F_t + K\|v\|_V^\alpha)\left(1 + \|v\|_H^\beta\right)\right)^{\frac{\alpha-1}{\alpha}}.$$

By Young's inequality we see that

$$\left((F_t + K\|v\|_V^\alpha)\left(1 + \|v\|_H^\beta\right)\right)^{\frac{\alpha-1}{\alpha}} \leqslant \frac{1}{\alpha} + \frac{\alpha-1}{\alpha}\left(F_t\left(1 + \|v\|_H^\beta\right) + K\|v\|_V^\alpha\left(1 + \|v\|_H^\beta\right)\right).$$

Therefore

$$\begin{aligned} \|B(t,v)\|_{L_2}^2 \leqslant\ & \frac{2}{\alpha} + F_t\left[1 + \frac{2(\alpha-1)}{\alpha}\left(1 + \|v\|_H^\beta\right)\right] \\ & + \|v\|_V^\alpha\left[K\frac{2(\alpha-1)}{\alpha}\left(1 + \|v\|_H^\beta\right) - \theta\right] + K\|v\|_H^2. \end{aligned}$$

Hence, since $\beta = 0$,

$$\|B(t,v)\|_{L_2}^2 \leqslant \frac{2}{\alpha} + F_t\left[1 + \frac{4(\alpha-1)}{\alpha}\right] + \|v\|_V^\alpha\left[K\frac{4(\alpha-1)}{\alpha} - \theta\right] + K\|v\|_H^2.$$

So, if $\theta + \gamma \geqslant 4K\frac{\alpha-1}{\alpha}$ *and if* $C \geqslant \max\left\{\frac{2}{\alpha}; K; 1 + 4\frac{\alpha-1}{\alpha}\right\}$, *then (A3) and (A4) give us a stronger estimate than (B1):*

$$\|B(t,v)\|_{L_2}^2 \leqslant C\left(1 + F_t + \|v\|_H^2\right) + \gamma\|v\|_V^\alpha.$$

Furthermore, since $\beta = 0$, *we can drop (B1) completely then, since (B2) covers the estimate for* f. *It remains to show that there exists such a* γ *with* $4K\frac{\alpha-1}{\alpha} - \theta \leqslant \gamma < \frac{\theta}{4}$. *By some calculation, this is true for* $5\theta < 16K$ *and*

$$\alpha < \left[1 - \frac{5\theta}{16K}\right]^{-1}.$$

The proof of Theorem 2.2.1 is split into an existence and a uniqueness part. The existence part is based on the so called Galerkin approximation. First we will consider equation (2.1.1) in a finite dimensional space with dimension $n \in \mathbb{N}$. Then a solution to this finite dimensional equation can be found, but instead of proving this fact, we refer to the literature. However, we will see that this solution fulfills some apriori estimates under our assumptions and this will lead to Lemma 2.3.7 below. There we will see that each integrand of (2.1.1) in the finite dimensional case convergences weakly as $n \to \infty$.
These limiting processes will be used to construct a solution to (2.1.1) in the general case. Section 2.3.2 deals with an Itô formula for this process and finally we will see that the integrands of our constructed process are almost everywhere equal to those given in (2.1.1). Hence a solution will be constructed, since all regularity estimates and integrability conditions will follow from conditions (A1)–(A4) and (B1)–(B3).
Section 2.4 deals with the matter of uniqueness. Contrary to the existence part, this one is quite easier. However, as already mentioned in the introduction, we will see that the stronger condition on γ in Theorem 2.2.1 (ii) is mandatory to obtain uniqueness of a solution.

Let us start with the proof of Theorem 2.2.1. For the rest of this chapter we set $p := \beta + 2$.

Notation. *For any given* $q > 1$ *we denote by* q' *its dual such that* $\frac{1}{q} + \frac{1}{q'} = 1$, *i.e.* $q' = \frac{q}{q-1}$.

We assume that for the initial value X_0 from Theorem 2.2.1 we have $X_0 \in L^{\beta+2}(\Omega, \mathcal{F}_0, P; H)$ without loss of generality. This follows from the generalized Hölder's inequality.

2.3. Proof of the Main Theorem – Existence

The proof of existence is based on the Galerkin approximation and therefore, we will first consider a finite dimensional version of equation (2.1.1).
Let $n \in \mathbb{N}$ be arbitrary. We will now assume that $\{e_1, e_2, \dots\} \subset V$ is an orthonormal basis of H, which exists since $V \subset H$ is dense and continuous and such that $\operatorname{Span}(e_1, e_2, \dots)$ is dense in V. Define the finite dimensional space

$$H_n := \operatorname{Span}(e_1, e_2, \dots, e_n)$$

and the projection

$$P_n\colon V^* \to H_n, \quad v \mapsto P_n(v) := \sum_{i=1}^{n} {}_{V^*}\langle v, e_i\rangle_V \, e_i.$$

For $u \in V$, $v \in H_n$ and $t \in [0,T]$ we obtain

$${}_{V^*}\langle P_n A(t,u), v\rangle_V = \langle P_n A(t,u), v\rangle_H = {}_{V^*}\langle A(t,u), v\rangle_V .$$

Now let $\{g_1, g_2, \dots\}$ be an orthonormal basis of U and $\tilde{P}_n$ the orthogonal projection onto $\mathrm{Span}(g_1, g_2, \dots, g_n)$ in U. Set

$$W_t^{(n)} := \sum_{i=1}^{n} \langle W_t, g_i\rangle_U \, g_i = \tilde{P}_n W_t.$$

2.3.1. Finite dimensional equation

The finite dimensional version of equation (2.1.1) in H_n can now be written as

$$\begin{aligned} \mathrm{d}Y(t) =& P_n A(t, Y(t))\,\mathrm{d}t + P_n B(t, Y(t))\,\mathrm{d}W_t^{(n)} \\ &+ \int_Z P_n f(t, Y(t-), z)\,\bar{\mu}(\mathrm{d}t, \mathrm{d}z), \\ Y(0) =& P_n X_0, \end{aligned} \tag{2.3.1}$$

where $t \in [0,T]$ and $X_0 \in L^{\beta+2}(\Omega, \mathcal{F}_t, P; H)$ is the same initial value as in Theorem 2.2.1.

2.3.1 Proposition. *Suppose conditions (A1)–(A4), (B1)–(B3) hold. Then equation (2.3.1) has a strong solution, i.e. there exists an $(\mathcal{F}_t)$-adapted, H_n-valued, càdlàg process $\left(X_t^{(n)}\right)_{t\in[0,T]}$ such that we have*

$$\begin{aligned} X_t^{(n)} =& P_n X_0 + \int_0^t P_n A\left(t, X_s^{(n)}\right)\mathrm{d}s + \int_0^t P_n B\left(s, X_s^{(n)}\right)\mathrm{d}W_s^{(n)} \\ &+ \int_0^t \int_Z P_n f\left(s, X_{s-}^{(n)}, z\right)\bar{\mu}(\mathrm{d}s, \mathrm{d}z). \end{aligned} \tag{2.3.2}$$

P-a.s. for all $t \in [0,T]$.

Proof. See [ABW10, Theorem 3.1]. □

2.3.2 Remark. *The result in Proposition 2.3.1 can also be retrieved from [GK80, Theorem 1].*

The next Lemma is an important auxiliary result and also known as the Itô formula.

2.3.3 Lemma (Itô's formula)**.** *Let $2 \leqslant q < \infty$ and, for fixed $n \in \mathbb{N}$, let $(X_t)_{t\in[0,T]}$ the stochastic process given in (2.3.2). Then*

$$
\begin{aligned}
\|X_t\|_H^q &= \|X_0\|_H^q + q\,(q-2)\int_0^t \|X_s\|_H^{q-4}\left\|\left(P_n B\,(s,X_s)\,\tilde{P}_n\right) * X_{s-}\right\|_H^2 \mathrm{d}s \\
&+\frac{q}{2}\int_0^t \|X_{s-}\|_H^{p-2}\left(2\,{}_V\langle A\,(s,X_s)\,,X_{s-}\rangle_{V^*} + \left\|P_n B\,(s,X_s)\,\tilde{P}_n\right\|_{L_2}^2\right)\mathrm{d}s \\
&+q\left(\int_0^t \|X_{s-}\|_H^{q-2}\langle X_{s-},P_n B\,(s,X_s)\,\mathrm{d}W_s\rangle_H + \int_0^t\int_Z \|X_{s-}\|_H^{q-2}\langle X_{s-},P_n f\,(s,X_{s-},z)\rangle_H\,\bar{\mu}\,(\mathrm{d}s,\mathrm{d}z)\right) \\
&+\int_0^t\int_Z\left(\|X_{s-}+P_n f\,(s,X_{s-},z)\|_H^q - \|X_{s-}\|_H^q - q\,\|X_{s-}\|_H^{q-2}\langle X_{s-},P_n f\,(s,X_s,z)\rangle_H\right)\mu\,(\mathrm{d}s,\mathrm{d}z)
\end{aligned}
$$

P-a.s. for all $t \in [0,T]$.

Proof. Apply [IW81, Theorem 5.1] to the function $x \mapsto \|x\|_H^p$ restricted to H_n. □

2.3.4 Remark. *Itô's formula – or Itô's lemma – for so called Itô-Lévy processes in finite dimensions is a well known result in the literature. It can also be found in [Mé82, Theorem 27.1] for general semimartingales. Without claiming to give a full list, let us further mention [App09, Theorem 4.4.7], [NØP09, Theorem 9.5] and [ABW10, Equation (2.16)].*

Now let us do some a priori estimates on $\left(X_t^{(n)}\right)_{t\in[0,T]}$ before we begin to construct a solution to (2.1.1). Recall that we set $p := \beta + 2$.

2.3.5 Lemma. *Suppose conditions (A1)–(A4) and (B1)–(B3) hold and that $F \in L^{\frac{p}{2}}([0,T]\times\Omega, \mathrm{d}t\times P)$. Let $\left(X_t^{(n)}\right)_{t\in[0,T]}$, $n\in\mathbb{N}$, be a solution to (2.3.1) given by Propositon 2.3.1.*

(i) There exists a constant $C_1 = C_1\left(p,\gamma,\theta,C,K,T,\|X_0\|_{L^p(\Omega;H)},\|F\|_{L^{\frac{p}{2}}(\Omega\times[0,T])}\right) > 0$ such that

$$
\sup_{t\in[0,T]}\mathbb{E}\left[\left\|X_t^{(n)}\right\|_H^2\right] + \mathbb{E}\left[\int_0^T\left\|X_t^{(n)}\right\|_V^\alpha \mathrm{d}t\right] \leqslant C_1 \tag{2.3.3}
$$

for all $n \in \mathbb{N}$.

(ii) There exists a constant $C_2 = C_2\,(p,\gamma,\theta,C,K,T) > 0$ such that

$$
\begin{aligned}
\sup_{t\in[0,T]}\mathbb{E}\left[\left\|X_t^{(n)}\right\|_H^p\right] &+ \mathbb{E}\left[\int_0^T\left\|X_t^{(n)}\right\|_H^{p-2}\left\|X_t^{(n)}\right\|_V^\alpha \mathrm{d}t\right] \\
&\leqslant C_2\left(\mathbb{E}\left[\|X_0\|_H^p\right] + \mathbb{E}\left[\int_0^T F_t^{\frac{p}{2}}\,\mathrm{d}t\right]\right)
\end{aligned} \tag{2.3.4}
$$

for all $n \in \mathbb{N}$.

(iii) *If* $0 \leqslant \gamma < \Gamma$*, then there exists a constant* $C_3 = C_3\left(p, \gamma, \theta, C, C_{BDG}, K, T\right) > 0$ *such that*

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|X_t^{(n)}\right\|_H^p\right] + \mathbb{E}\left[\int_0^T \left\|X_t^{(n)}\right\|_H^{p-2}\left\|X_t^{(n)}\right\|_V^{\alpha}\,\mathrm{d}t\right] \\ \leqslant C_3\left(\mathbb{E}\left[\|X_0\|_H^p\right] + \mathbb{E}\left[\int_0^T F_t^{\frac{p}{2}}\,\mathrm{d}t\right]\right)\end{aligned} \tag{2.3.5}$$

for all $n \in \mathbb{N}$*.*

2.3.6 Remark.

(i) *In the proof of 2.3.5 (ii) and (iii) (and also Lemma 2.3.7 (ii) to (iv)) we heavily use Young's inequality with* $q = \frac{p}{p-2}$*. At first sight this is not possible in case that* $p = 2$*, i.e.* $\beta = 0$*. But we use it always in the same situation, namely*

$$\xi^{p-2}\zeta \leqslant \frac{p-2}{p}\xi^p + \frac{2}{p}\zeta^{\frac{p}{2}}$$

for $\xi, \zeta \in \mathbb{R}$*, and this inequality holds true even if* $p = 2$*.*

(ii) *One may also note that 2.3.5 (i) and (ii) are identical if* $p = 2$*. But we can see in steps (i).6 and (ii).7 of the proof that 2.3.5 (i) allows us to use a higher bound, namely* $\gamma < \theta$ *instead of* $\gamma < \frac{1}{4}\theta$ *as in the proof of (ii). This is because in the proof of (i) we use the second part from Lemma C.1 and not the first part.*

Proof. First we need to introduce a stopping time $\tau_R^{(n)}$ for given $n \in \mathbb{N}$ and $R > 0$, defined by

$$\tau_R^{(n)} = \inf\left\{t \geqslant 0 \,\middle|\, \left\|X_t^{(n)}\right\|_H > R\right\} \wedge T.$$

By Theorem D.4 we know that $\tau_R^{(n)}$ is a stopping time. Furthermore we have $\lim_{R\to\infty}\tau_R^{(n)} = T$ P-a.s. Since $X_t^{(n)}$ takes values in the finite dimensional space $H_n \subset V$ and because $V \subset H$ continuously, we have

$$\left\|X_t^{(n)}\right\|_H \leqslant R, \qquad \left\|X_t^{(n)}\right\|_V \leqslant \bar{C}R, \qquad \text{for all } t \in \left[0, \tau_R^{(n)}\right],\ n \in \mathbb{N}. \tag{2.3.6}$$

Notation. *To avoid notational complexity we set* $\hat{t} := \hat{t}\,(t, n, R) := t \wedge \tau_R^{(n)}$ *for* $t \in [0, T]$*.*

Step (i).1. Let us apply Itô's formula 2.3.3 to the process $X_t^{(n)}$ and for $q = 2$ in there.

Then P-a.s. for all $t \in [0,T]$ we have

$$\begin{aligned}\left\|X_t^{(n)}\right\|_H^2 = & \left\|X_0^{(n)}\right\|_H^2 + \int_0^t \left(2\,{}_{V^*}\!\left\langle A\left(s, X_s^{(n)}\right), X_{s-}^{(n)}\right\rangle_V + \left\|P_n B\left(s, X_s^{(n)}\right)\tilde{P}_n\right\|_{L_2}^2\right) \mathrm{d}s \\ & + 2\int_0^t \left\langle X_{s-}^{(n)}, P_n B\left(s, X_s^{(n)}\right) \mathrm{d}W_s^{(n)}\right\rangle_H \\ & + 2\int_0^t \int_Z \left\langle X_{s-}^{(n)}, P_n f\left(s, X_{s-}^{(n)}, z\right)\right\rangle_H \bar{\mu}(\mathrm{d}s, \mathrm{d}z) \\ & + \int_0^t \int_Z \left(\left\|X_{s-}^{(n)} + P_n f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^2 - \left\|X_{s-}^{(n)}\right\|_H^2\right) \mu(\mathrm{d}s, \mathrm{d}z) \\ & - 2\int_0^t \int_Z \left\langle X_{s-}^{(n)}, P_n f\left(s, X_{s}^{(n)}, z\right)\right\rangle_H \mu(\mathrm{d}s, \mathrm{d}z) \\ =: & \left\|X_0^{(n)}\right\|_H^2 + H_1(t) + 2H_2(t) + 2H_3(t) + H_4(t) - 2H_5(t).\end{aligned} \tag{2.3.7}$$

Step (i).2. Applying (A3) to $H_1(\hat{t})$ yields to

$$\begin{aligned}H_1(\hat{t}) &= \int_0^{\hat{t}} \left(2\,{}_{V^*}\!\left\langle A\left(s, X_s^{(n)}\right), X_{s-}^{(n)}\right\rangle_V + \left\|P_n B\left(s, X_s^{(n)}\right)\tilde{P}_n\right\|_{L_2}^2\right) \mathrm{d}s \\ &\overset{(A3)}{\leqslant} \int_0^{\hat{t}} F_s \,\mathrm{d}s + K\int_0^{\hat{t}} \left\|X_s^{(n)}\right\|_H^2 \mathrm{d}s - \theta \int_0^{\hat{t}} \left\|X_s^{(n)}\right\|_V^\alpha \mathrm{d}s.\end{aligned}$$

Step (i).3. We come to $H_2(\hat{t})$. Let us note that we have, for all $t \in [0,T]$ and $v \in V$,

$$\|B(t,v)\|_{L_2}^2 \leqslant \|B(t,v)\|_{L_2}^2 + \int_Z \|f(t,v,z)\|_H^2 \, m(\mathrm{d}z) \overset{(B1)}{\leqslant} C\left(1 + F_t + \|v\|_H^2\right) + \gamma \|v\|_V^\alpha. \tag{2.3.8}$$

The stochastic integral $\int_0^{\hat{t}} \left\langle X_{s-}^{(n)}, P_n B\left(s, X_s^{(n)}\right) \mathrm{d}W_s^{(n)}\right\rangle_H$ is well-defined as a contiuous, real-valued, local martingale, since as a càdlàg, $\mathcal{F}_t$-adapted process $X_{t-}^{(n)}$ is predictable and

$$\mathbb{E}\left[\int_0^{\hat{t}} \left\|P_n B\left(s, X_s^{(n)}\right)\right\|_{L_2}^2 \mathrm{d}s\right] \overset{(2.3.8)}{\leqslant} \mathbb{E}\left[\int_0^{\hat{t}} \left(C\left(1 + F_t + \left\|X_s^{(n)}\right\|_H^2\right) + \gamma \left\|X_s^{(n)}\right\|_V^\alpha\right) \mathrm{d}s\right] < \infty,$$

because from (2.3.6) we see that $\left\|X_s^{(n)}\right\|_H^2 < \infty$ and $\left\|X_s^{(n)}\right\|_V^\alpha < \infty$ and we have $F \in L^1([0,T] \times \Omega, \mathrm{d}t \otimes P; \mathbb{R})$. Hence we deduce $\mathbb{E}\left[H_2(\hat{t})\right] = 0$.

Step (i).4. Let us show that $\mathbb{E}\left[H_3(\hat{t})\right] = 0$. Let $\Phi(s,z) := \left\langle X_{s-}^{(n)}, P_n f\left(s, X_{s-}^{(n)}, z\right)\right\rangle_H$, then the process $s \mapsto \Phi(s, \cdot)$ is predictable, since f is predictable. From Proposition 1.2.2

and 1.2.3 we deduce that $H_3\left(\hat{t}\right)$ is a martingale. Indeed, for all $t \in [0,T]$ and $v \in V$, we have by the Cauchy-Schwarz inequality and condition (B1)

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\hat{t}}\int_Z \left|\left\langle X_{s-}^{(n)}, P_n f\left(s, X_{s-}^{(n)}, z\right)\right\rangle_H\right|^2 m\left(\mathrm{d}z\right)\,\mathrm{d}s\right] \\
&\leqslant \mathbb{E}\left[\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^2 \int_Z \left\|P_n f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^2 m\left(\mathrm{d}z\right)\,\mathrm{d}s\right] \\
&\overset{\text{(B1)}}{\leqslant} \mathbb{E}\left[\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^2 \left(C\left(1 + F_s + \left\|X_{s-}^{(n)}\right\|_H^2\right) + \gamma\left\|X_{s-}^{(n)}\right\|_V^\alpha\right)\mathrm{d}s\right] < \infty,
\end{aligned}$$

because by (2.3.6) the V- and H-norms of $X_{s-}^{(n)}$ are bounded and $F_s \in L^1\left([0,T]\times\Omega, \mathrm{d}t\otimes P;\mathbb{R}\right)$.
Step (i).5. As the last step of preparation we want to estimate $\mathbb{E}\left[H_4\left(\hat{t}\right) - 2H_5\left(\hat{t}\right)\right]$. Proposition 1.2.7 allows us to change the integrator from $\mu\left(\mathrm{d}s, \mathrm{d}z\right)$ to $m\left(\mathrm{d}z\right)\mathrm{d}s$ and then we apply Lemma C.1:

$$\begin{aligned}
\mathbb{E}\left[H_4\left(\hat{t}\right) - 2H_5\left(\hat{t}\right)\right] \leqslant\; & \left|\mathbb{E}\left[H_4\left(\hat{t}\right) - 2H_5\left(\hat{t}\right)\right]\right| \leqslant \mathbb{E}\left[\left|H_4\left(\hat{t}\right) - 2H_5\left(\hat{t}\right)\right|\right] \\
\leqslant\; & \mathbb{E}\left[\int_0^t\int_Z \left|\left\|X_{s-}^{(n)} + P_n f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^2 - \left\|X_{s-}^{(n)}\right\|_H^2\right.\right. \\
& \left.\left. - 2\left\langle X_{s-}^{(n)}, P_n f\left(s, X_s^{(n)}, z\right)\right\rangle_H\right| \mu\left(\mathrm{d}s, \mathrm{d}z\right)\right] \\
\overset{1.2.7}{=}\; & \mathbb{E}\left[\int_0^t\int_Z \left|\left\|X_{s-}^{(n)} + P_n f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^2 - \left\|X_{s-}^{(n)}\right\|_H^2\right.\right. \\
& \left.\left. - 2\left\langle X_{s-}^{(n)}, P_n f\left(s, X_s^{(n)}, z\right)\right\rangle_H\right| m\left(\mathrm{d}z\right)\,\mathrm{d}s\right] \\
\overset{C.1}{=}\; & \mathbb{E}\left[\int_0^{\hat{t}}\int_Z \left\|f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^2 m\left(\mathrm{d}z\right)\,\mathrm{d}s\right].
\end{aligned}$$

Therefore, by (B1), we know that, for all $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[H_4\left(\hat{t}\right) - 2H_5\left(\hat{t}\right)\right] \leqslant \mathbb{E}\left[\int_0^{\hat{t}}\int_Z \left\|f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^2 m\left(\mathrm{d}z\right)\,\mathrm{d}s\right] \\
&\overset{\text{(B1)}}{\leqslant} CT + C\,\mathbb{E}\left[\int_0^{\hat{t}} F_s\,\mathrm{d}s\right] + C\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^2\,\mathrm{d}s\right] + \gamma\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_V^\alpha\,\mathrm{d}s\right].
\end{aligned}$$

Step (i).6. The results (i).2 to (i).5 combined and used in the stopped version of (2.3.7) in expectation delivers, for all $t \in [0,T]$,

$$\begin{aligned}\mathbb{E}\left[\left\|X_{\hat{t}}^{(n)}\right\|_H^2\right] =& \mathbb{E}\left[\left\|X_0^{(n)}\right\|_H^2\right] + \mathbb{E}\left[H_1\left(\hat{t}\right)\right] + 2\underbrace{\mathbb{E}\left[H_2\left(\hat{t}\right)\right]}_{=0} + 2\underbrace{\mathbb{E}\left[H_3\left(\hat{t}\right)\right]}_{=0} + \mathbb{E}\left[H_4\left(\hat{t}\right) - 2H_5\left(\hat{t}\right)\right]\\ \leqslant& \mathbb{E}\left[\|X_0\|_H^2\right] + (1+C)\,\mathbb{E}\left[\int_0^T F_s\,\mathrm{d}s\right] + (C+K)\,\mathbb{E}\left[\int_0^{\hat{t}} \left\|X_s^{(n)}\right\|_H^2 \mathrm{d}s\right]\\ &+ CT + (\gamma-\theta)\,\mathbb{E}\left[\int_0^{\hat{t}} \left\|X_s^{(n)}\right\|_V^\alpha \mathrm{d}s\right].\end{aligned} \tag{2.3.9}$$

Here we used that, since $X_0^{(n)}$ is H_n-valued, we have $\left\|X_0^{(n)}\right\|_H \leqslant \|X_0\|_H$ and that, because F is non-negative and hence the integral with respect to ds is increasing in time, we get $\mathbb{E}\left[\int_0^{\hat{t}} F_s\,\mathrm{d}s\right] \leqslant \mathbb{E}\left[\int_0^T F_s\,\mathrm{d}s\right]$. If $p > 2$ then we observe by Hölder's inequality

$$\mathbb{E}\left[\|X_0\|_H^2\right] \leqslant \mathbb{E}\left[\|X_0\|_H^p\right]^{\frac{2}{p}} \cdot \mathbb{E}\left[1\right]^{\frac{p-2}{p}} = \mathbb{E}\left[\|X_0\|_H^p\right]^{\frac{2}{p}} < \infty$$

and

$$(1+C)\,\mathbb{E}\left[\int_0^T F_s\,\mathrm{d}s\right] \leqslant (1+C)\,T^{\frac{p-2}{p}}\,\mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right]^{\frac{2}{p}} < \infty.$$

So we set $\Theta := \mathbb{E}\left[\|X_0\|_H^p\right]^{\frac{2}{p}} + (1+C)\,T^{\frac{p-2}{p}}\,\mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right]^{\frac{2}{p}} + CT$. We also have $\Theta < \infty$. In the case $p = 2$ we see immediately that $\Theta = \mathbb{E}\left[\|X_0\|_H^2\right] + 2\,\mathbb{E}\left[\int_0^T F_s\,\mathrm{d}s\right] + CT < \infty$ by assumption. Since $\gamma < \theta\frac{p}{2}\left[p\,(p-1) + 2^{p-1}\,(2p-3)\right]^{-1} \overset{p\geqslant 2}{\leqslant} \theta\frac{p}{2p} < \theta$ by assumption, we have $\theta - \gamma > 0$ and we bring the last summand in (2.3.9) to the left hand side and get by Fubini's theorem

$$\varphi\left(\hat{t}\right) + (\theta-\gamma)\,\psi\left(\hat{t}\right) \leqslant \Theta + \int_0^{\hat{t}} (C+K)\,\varphi(s)\;\mathrm{d}s$$

with $\varphi(t) = \mathbb{E}\left[\left\|X_t^{(n)}\right\|_H^2\right]$, $\psi(t) = \mathbb{E}\left[\left\|X_s^{(n)}\right\|_V^\alpha\right]$. Furthermore we have $C + K > 0$ and $\psi \geqslant 0$. Therefore we can apply Gronwall's lemma B.5 on

$$\varphi\left(\hat{t}\right) + (\theta-\gamma)\,\psi\left(\hat{t}\right) \leqslant \Theta + \int_0^{\hat{t}} (C+K)\,(\varphi(s) + (\theta-\gamma)\,\psi(s))\;\mathrm{d}s$$

and we get

$$\varphi\left(\hat{t}\right) + (\theta-\gamma)\,\psi\left(\hat{t}\right) \leqslant \Theta e^{(C+K)\hat{t}} \leqslant \Theta e^{(C+K)T}.$$

Then $\varphi\left(\hat{t}\right) \leqslant \Theta e^{(C+K)T}$ and hence $\sup_{s\in[0,\hat{t}]}\varphi(s) \leqslant \Theta e^{(C+K)T}$.

$$\psi\left(\hat{t}\right) \leqslant \frac{1}{\theta-\gamma}\left(\varphi\left(\hat{t}\right) + (\theta-\gamma)\,\psi\left(\hat{t}\right)\right) \leqslant \frac{\Theta}{\theta-\gamma}e^{(C+K)T}$$

also holds true. Resubstitution gives us

$$\begin{aligned}
&\sup_{s\in[0,\hat{t}]} \mathbb{E}\left[\left\|X_s^{(n)}\right\|_H^2\right] + \mathbb{E}\left[\left\|X_{\hat{t}}^{(n)}\right\|_V^\alpha\right] \\
&\leqslant \left(1+\frac{1}{\theta-\gamma}\right) e^{(C+K)T} \left(\mathbb{E}\left[\|X_0\|_H^p\right]^{\frac{2}{p}} + 2T^{\frac{p-2}{p}}\,\mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right]^{\frac{2}{p}} + CT\right) := C_1
\end{aligned} \tag{2.3.10}$$

for all $t \in [0,T]$ and $n \in \mathbb{N}$ and with $C_1 = C_1\left(p, \gamma, \theta, C, K, T, \|X_0\|_{L^p(\Omega;H)}, \|F\|_{L^{\frac{p}{2}}(\Omega\times[0,T])}\right) < \infty$. The right hand side is independent of t, R, n and the stopping time $\tau_R^{(n)}$.
Step (i).7. Now, we apply the monotone convergence theorem. (2.3.10) holds for $T \in [0,T]$ and we have $\tau_R^{(n)} \to T$ as $R \to \infty$ P-a.s. Then we have

$$\begin{aligned}
&\sup_{s\in[0,T]} \mathbb{E}\left[\left\|X_s^{(n)}\right\|_H^2\right] + \mathbb{E}\left[\left\|X_T^{(n)}\right\|_V^\alpha\right] \\
&= \lim_{R\to\infty} \sup_{s\in\left[0,T\wedge\tau_R^{(n)}\right]} \mathbb{E}\left[\left\|X_s^{(n)}\right\|_H^2\right] + \mathbb{E}\left[\lim_{R\to\infty}\left\|X_{T\wedge\tau_R^{(n)}}^{(n)}\right\|_V^\alpha\right] \\
&= \lim_{R\to\infty}\left(\sup_{s\in\left[0,T\wedge\tau_R^{(n)}\right]} \mathbb{E}\left[\left\|X_s^{(n)}\right\|_H^2\right] + \mathbb{E}\left[\left\|X_{T\wedge\tau_R^{(n)}}^{(n)}\right\|_V^\alpha\right]\right) \\
&\overset{(2.3.10)}{\leqslant} C_1
\end{aligned}$$

for all $n \in \mathbb{N}$.
We come to the proof of 2.3.5 (ii).
Step (ii).1. We apply Itô's formula 2.3.3 to the process $X_t^{(n)}$ with $q = p$. Then P-a.s. for

all $t \in [0,T]$ we have

$$\begin{aligned}
\left\|X_t^{(n)}\right\|_H^p = & \left\|X_0^{(n)}\right\|_H^p + p(p-2)\int_0^t \left\|X_{s-}^{(n)}\right\|_H^{p-4} \left\|\left(P_n B\left(s, X_s^{(n)}\right)\tilde{P}_n\right) * X_{s-}^{(n)}\right\|_H^2 \mathrm{d}s \\
& + \frac{p}{2}\int_0^t \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left(2\,{}_{V^*}\left\langle A\left(s, X_s^{(n)}\right), X_{s-}^{(n)}\right\rangle_V + \left\|P_n B\left(s, X_s^{(n)}\right)\tilde{P}_n\right\|_{L_2}^2\right) \mathrm{d}s \\
& + p\int_0^t \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\langle X_{s-}^{(n)}, P_n B\left(s, X_s^{(n)}\right) \mathrm{d}W_s^{(n)}\right\rangle_H \\
& + p\int_0^t\int_Z \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\langle X_{s-}^{(n)}, P_n f\left(s, X_{s-}^{(n)}, z\right)\right\rangle_H \bar{\mu}(\mathrm{d}s, \mathrm{d}z) \\
& + \int_0^t\int_Z \left(\left\|X_{s-}^{(n)} + P_n f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^p - \left\|X_{s-}^{(n)}\right\|_H^p\right) \mu(\mathrm{d}s, \mathrm{d}z) \\
& - p\int_0^t\int_Z \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\langle X_{s-}^{(n)}, P_n f\left(s, X_s^{(n)}, z\right)\right\rangle_H \mu(\mathrm{d}s, \mathrm{d}z) \\
=: & \left\|X_0^{(n)}\right\|_H^p + p(p-2) I_1(t) + \frac{p}{2} I_2(t) + pI_3(t) + pI_4(t) + I_5(t) - pI_6(t).
\end{aligned} \tag{2.3.11}$$

Step (ii).2. We use (2.3.8) for $I_1\left(\hat{t}\right)$.

$$\begin{aligned}
I_1\left(\hat{t}\right) = & \int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-4} \left\|\left(P_n B\left(s, X_s^{(n)}\right)\tilde{P}_n\right) * X_{s-}^{(n)}\right\|_H^2 \mathrm{d}s \\
\leqslant & \int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-4} \left\|\left(P_n B\left(s, X_s^{(n)}\right)\tilde{P}_n\right)\right\|_H^2 \left\|X_s^{(n)}\right\|_H^2 \mathrm{d}s \\
\overset{(2.3.8)}{\leqslant} & \int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left[C\left(1 + F_s + \left\|X_s^{(n)}\right\|_H^2\right) + \gamma\left\|X_s^{(n)}\right\|_V^\alpha\right] \mathrm{d}s \\
= & C\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} F_s \,\mathrm{d}s + C\int_0^t \left\|X_{s-}^{(n)}\right\|_H^p \mathrm{d}s + \gamma\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\|X_s^{(n)}\right\|_V^\alpha \mathrm{d}s \\
& + C\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} \mathrm{d}s.
\end{aligned}$$

The first summand in the last line can be splitted by Young's inequality with $q = \frac{p}{p-2}$ into $\frac{p-2}{p}\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^p \mathrm{d}s + \frac{2}{p}C^{\frac{p}{2}}\int_0^{\hat{t}} F_s^{\frac{p}{2}} \mathrm{d}s$ and the last summand into $\frac{2T}{p}C^{\frac{p}{2}} + \frac{p-2}{p}\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^p \mathrm{d}s$. Then we obtain

$$I_1\left(\hat{t}\right) \leqslant \gamma\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\|X_s^{(n)}\right\|_V^\alpha \mathrm{d}s + \left(C + \frac{2(p-2)}{p}\right)\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^p \mathrm{d}s + \frac{2}{p}C^{\frac{p}{2}}\int_0^{\hat{t}} F_s^{\frac{p}{2}} \mathrm{d}s + \frac{2T}{p}C^{\frac{p}{2}}.$$

Step (ii).3. We apply (A3) to $I_2\left(\hat{t}\right)$.

$$\begin{aligned} I_2\left(\hat{t}\right) &= \int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left(2\,{}_{V^*}\!\left\langle A\left(s, X_s^{(n)}\right), X_{s-}^{(n)}\right\rangle_V + \left\|P_n B\left(s, X_s^{(n)}\right)\tilde{P}_n\right\|_{L_2}^2\right) \mathrm{d}s \\ &\overset{(A3)}{\leqslant} \int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left(F_s + K\left\|X_{s-}^{(n)}\right\|_H^2 - \theta\left\|X_{s-}^{(n)}\right\|_V^\alpha\right) \mathrm{d}s \\ &= \int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} F_s\,\mathrm{d}s + K\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^p \mathrm{d}s - \theta\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\|X_{s-}^{(n)}\right\|_V^\alpha \mathrm{d}s. \end{aligned}$$

As in the step before we split the first summand of the last line by Young's inequality and have the following result

$$I_2\left(\hat{t}\right) \leqslant -\theta\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\|X_{s-}^{(n)}\right\|_V^\alpha \mathrm{d}s + \left(K + \frac{p-2}{p}\right)\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^p \mathrm{d}s + \frac{2}{p}\int_0^{\hat{t}} F_s^{\frac{p}{2}}\,\mathrm{d}s.$$

Step (ii).4. Let us show that $\Phi\left(s,z\right) := \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\langle X_{s-}^{(n)}, P_n f\left(s, X_{s-}^{(n)}, z\right)\right\rangle_H \in \mathcal{N}_{\bar{\mu}}^2\left(\hat{t}, Z; \mathbb{R}\right)$, then $I_4\left(\hat{t}\right) = \int_0^{\hat{t}}\int_Z \Phi\left(s,z\right)\bar{\mu}\left(\mathrm{d}s,\mathrm{d}z\right)$ is a real-valued martingale by Proposition 1.2.2 and 1.2.3 and we get $\mathbb{E}\left[I_4\left(\hat{t}\right)\right] = \mathbb{E}\left[\int_0^{\hat{t}}\int_Z \Phi\left(s,z\right)\bar{\mu}\left(\mathrm{d}s,\mathrm{d}z\right)\right] = 0$. Since f is predictable, the process $s \mapsto \Phi\left(s,\cdot\right)$ is predictable. It remains to show that $\|\Phi\|_{\hat{t}} < \infty$. By condition (B1) we get for all $t \in [0,T]$ and $v \in V$

$$\int_Z \left\|f\left(t,v,z\right)\right\|_H^2 m\left(\mathrm{d}z\right) \overset{(B1)}{\leqslant} C\left(1 + F_t + \|v\|_H^2\right) + \gamma\|v\|_V^\alpha \tag{2.3.12}$$

and the Cauchy-Schwarz inequality delivers

$$\begin{aligned} \|\Phi\|_{\hat{t}}^2 &= \mathbb{E}\left[\int_0^{\hat{t}}\int_Z \left|\left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\langle X_{s-}^{(n)}, P_n f\left(s, X_{s-}^{(n)}, z\right)\right\rangle_H\right|^2 m\left(\mathrm{d}z\right)\mathrm{d}s\right] \\ &\overset{C.-S.}{\leqslant} \mathbb{E}\left[\int_0^{\hat{t}}\int_Z \left\|X_{s-}^{(n)}\right\|_H^{2p-4} \left\|X_{s-}^{(n)}\right\|_H^2 \left\|P_n f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^2 m\left(\mathrm{d}z\right)\mathrm{d}s\right] \\ &= \mathbb{E}\left[\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{2(p-1)} \int_Z \left\|P_n f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^2 m\left(\mathrm{d}z\right)\mathrm{d}s\right] \\ &\overset{(2.3.12)}{\leqslant} \mathbb{E}\left[\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{2(p-1)} \left(C\left(1 + F_s + \left\|X_{s-}^{(n)}\right\|_H^2\right) + \gamma\left\|X_{s-}^{(n)}\right\|_V^\alpha\right)\mathrm{d}s\right] \\ &< \infty, \end{aligned}$$

since $\left\|X_{s-}^{(n)}\right\|_V$ and $\left\|X_{s-}^{(n)}\right\|_H$ are bounded by (2.3.6) and $F \in L^{\frac{p}{2}}\left([0,T]\times\Omega, \mathrm{d}t\times P\right)$.

Step (ii).5. Since, as a càdlàg, $\mathcal{F}_t$-adapted process, $X_{t-}^{(n)}$ is predictable, the stochastic integral $\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\langle X_{s-}^{(n)}, P_n B\left(s, X_s^{(n)}\right)\mathrm{d}W_s^{(n)}\right\rangle$ is well-defined as a continuous, real-valued, local martingale if $\Phi(s) := P_n B\left(s, X_s^{(n)}\right) \in \mathcal{N}_{W^{(n)}}\left(0, \hat{t}\right)$.

$$\begin{aligned} \|\Phi\|_{\hat{t}} = & \mathbb{E}\left[\int_0^{\hat{t}} \left\|P_n B\left(s, X_s^{(n)}\right)\right\|_{L_2}^2 \mathrm{d}s\right] \\ \overset{(2.3.8)}{\leqslant} & \mathbb{E}\left[\int_0^{\hat{t}} \left(C\left(1 + F_t + \left\|X_s^{(n)}\right\|_H^2\right) + \gamma \left\|X_s^{(n)}\right\|_V^\alpha\right) \mathrm{d}s\right] \\ < & \infty \end{aligned}$$

as in Step (ii).4. Hence we have $\mathbb{E}\left[I_3\left(\hat{t}\right)\right] = 0$.
Step (ii).6. Now let us come to $I_5\left(\hat{t}\right)$ and $I_6\left(\hat{t}\right)$. First, we want to estimate $\mathbb{E}\left[\left|I_5\left(\hat{t}\right) - pI_6\left(\hat{t}\right)\right|\right]$ by Lemma C.1. By Proposition 1.2.7 we can replace $\mu(\mathrm{d}s, \mathrm{d}z)$ by $m(\mathrm{d}z)\,\mathrm{d}s$ in the integral and then there exists a constant $C_4 = C_4(p)$ such that

$$\begin{aligned} \mathbb{E}\left[\left|I_5\left(\hat{t}\right) - pI_6\left(\hat{t}\right)\right|\right] \leqslant & \mathbb{E}\left[\int_0^{\hat{t}} \int_Z \left| \left\|X_{s-}^{(n)} + P_n f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^p - \left\|X_{s-}^{(n)}\right\|_H^p \right.\right. \\ & \left.\left. -p \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\langle X_{s-}^{(n)}, P_n f\left(s, X_s^{(n)}, z\right)\right\rangle_H \right| \mu(\mathrm{d}s, \mathrm{d}z)\right] \\ \overset{1.2.7}{=} & \mathbb{E}\left[\int_0^{\hat{t}} \int_Z \left| \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\|X_{s-}^{(n)} + P_n f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^p - \left\|X_{s-}^{(n)}\right\|_H^p \right.\right. \\ & \left.\left. -p \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\langle X_{s-}^{(n)}, P_n f\left(s, X_s^{(n)}, z\right)\right\rangle_H \right| m(\mathrm{d}z)\,\mathrm{d}s\right] \\ \overset{C.1}{\leqslant} & C_4\, \mathbb{E}\left[\int_0^{\hat{t}} \int_Z \left(\left\|X_{s-}^{(n)}\right\|_H^p + \left\|P_n f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^p\right) m(\mathrm{d}z)\,\mathrm{d}s\right]. \end{aligned}$$

Continuing in the last row of the equation above and using (B2) we now obtain

$$\begin{aligned} & \mathbb{E}\left[\left|I_5\left(\hat{t}\right) - pI_6\left(\hat{t}\right)\right|\right] \\ \leqslant & C_4\, \mathbb{E}\left[\int_0^{\hat{t}} \left(\left\|X_{s-}^{(n)}\right\|_H^p + C\left(1 + F_s^{\frac{p}{2}} + \left\|X_{s-}^{(n)}\right\|_H^p\right) + \gamma \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\|X_{s-}^{(n)}\right\|_V^\alpha\right) \mathrm{d}s\right] \\ \leqslant & C_4(C+1)\, \mathbb{E}\left[\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^p \mathrm{d}s\right] + C \cdot C_4\, \mathbb{E}\left[\int_0^{\hat{t}} F_s^{\frac{p}{2}} \mathrm{d}s\right] \\ & + \gamma C_4\, \mathbb{E}\left[\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\|X_{s-}^{(n)}\right\|_V^\alpha \mathrm{d}s\right] + C \cdot C_4 T. \end{aligned}$$

Step (ii).7. We apply the expectation to the stopped version of (2.3.11) and use our results from the Steps (ii).2 to (ii).6 on it.

$$\begin{aligned}
\mathbb{E}\left[\left\|X_{\hat{t}}^{(n)}\right\|_H^p\right] =& \mathbb{E}\left[\|X_0\|_H^p\right] + p\,(p-2)\,\mathbb{E}\left[I_1\left(\hat{t}\right)\right] + \frac{p}{2}\,\mathbb{E}\left[I_2\left(\hat{t}\right)\right] + p\underbrace{\mathbb{E}\left[I_3\left(\hat{t}\right)\right]}_{=0} + p\underbrace{\mathbb{E}\left[I_4\left(\hat{t}\right)\right]}_{=0} \\
&+ \underbrace{\mathbb{E}\left[I_5\left(\hat{t}\right) - pI_6\left(\hat{t}\right)\right]}_{\leqslant\left|\mathbb{E}\left[I_5\left(\hat{t}\right)-pI_6\left(\hat{t}\right)\right]\right|\leqslant\mathbb{E}\left[\left|I_5\left(\hat{t}\right)-pI_6\left(\hat{t}\right)\right|\right]} \\
&\leqslant \mathbb{E}\left[\|X_0\|_H^p\right] + \left(\left(p\,(p-2)+C_4\right)\gamma - \frac{p}{2}\theta\right)\mathbb{E}\left[\int_0^{\hat{t}} \left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right] \\
&+ C_5\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^p\,\mathrm{d}s\right] + C_6\,\mathbb{E}\left[\int_0^{\hat{t}} F_s^{\frac{p}{2}}\,\mathrm{d}s\right] + \left(C\cdot C_4 + 2\,(p-2)\,C^{\frac{p}{2}}\right)T,
\end{aligned}$$

where

$$C_5 := \left(p\,(p-2)\left(C + \frac{2\,(p-2)}{p}\right) + \frac{p}{2}\left(K + \frac{p-2}{p}\right) + C_4\,(C+1)\right) \quad \text{and}$$
$$C_6 := 2\,(p-2)\,C^{\frac{p}{2}} + C\cdot C_4 + 1.$$

Here we used that since $X_0^{(n)}$ is H_n-valued we have $\left\|X_0^{(n)}\right\|_H \leqslant \|X_0\|_H$. Bringing the second summand to the left side we get

$$\begin{aligned}
\mathbb{E}\left[\left\|X_{\hat{t}}^{(n)}\right\|_H^p\right] + &\left(\frac{p}{2}\theta - \left(p\,(p-2)+C_4\right)\gamma\right)\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_{s-}^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right] \\
\leqslant&\ \mathbb{E}\left[\|X_0\|_H^p\right] + C_5\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^p\,\mathrm{d}s\right] + C_6\,\mathbb{E}\left[\int_0^{\hat{t}} F_s^{\frac{p}{2}}\,\mathrm{d}s\right] \\
&+ \left(C\cdot C_4 + 2\,(p-2)\,C^{\frac{p}{2}}\right)T,
\end{aligned}$$

where $\left(p\,(p-2)+C_4\right)\gamma < \frac{p}{2}\theta$ by assumption on γ. For simplicity we set $\varphi\,(t) := \mathbb{E}\left[\left\|X_t^{(n)}\right\|_H^p\right]$, $\psi\,(t) := \mathbb{E}\left[\int_0^t \left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_{s-}^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right]$ and $C_7 = C_7\,(p,\gamma,\theta) = \frac{p}{2}\theta - \left(p\,(p-2)+C_4\right)\gamma > 0$. We want to apply Gronwall's inequality, therefore by Fubini's theorem the inequality above can be written as

$$\varphi\left(\hat{t}\right) + C_7\psi\left(\hat{t}\right) \leqslant \Theta + \int_0^{\hat{t}} C_5\varphi\,(s)\,\mathrm{d}s,$$

where $\Theta := \mathbb{E}\left[\|X_0\|_H^p\right] + C_6\,\mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right] + \left(C\cdot C_4 + 2\,(p-2)\,C^{\frac{p}{2}}\right)T$. Note that $C_5 > 0$ because $K > 0$. Since ψ is non-negative, we focus on

$$\varphi\left(\hat{t}\right) + C_7\psi\left(\hat{t}\right) \leqslant \Theta + \int_0^{\hat{t}} C_5\varphi\,(s)\,\mathrm{d}s \leqslant \Theta + \int_0^{\hat{t}} C_5\left(\varphi\,(s) + C_7\psi\,(s)\right)\mathrm{d}s$$

and Gronwall's inequality B.5 gives us

$$\varphi\left(\hat{t}\right)+C_7\psi\left(\hat{t}\right)\leqslant\Theta e^{C_5\hat{t}}\leqslant\Theta e^{C_5T}.$$

Non-negativity and $C_7>0$ gives us $\varphi\left(\hat{t}\right)\leqslant\varphi\left(\hat{t}\right)+C_7\psi\left(\hat{t}\right)\leqslant\Theta e^{C_5T}$, therefore $\sup_{r\in[0,\hat{t}]}\varphi\left(r\right)\leqslant\Theta e^{C_5T}$, and

$$\psi\left(\hat{t}\right)\leqslant\frac{1}{C_7}\left(\varphi\left(\hat{t}\right)+C_7\psi\left(\hat{t}\right)\right)\leqslant\frac{1}{C_7}\Theta e^{C_5T}.$$

Altogether we have (with resubstituting Θ) for all $t\in[0,T]$ and $n\in\mathbb{N}$

$$\begin{aligned}&\sup_{r\in[0,\hat{t}]}\varphi\left(r\right)+\psi\left(\hat{t}\right)\\&\leqslant\left(1+\frac{1}{C_7}\right)e^{C_5T}\left(\mathbb{E}\left[\|X_0\|_H^p\right]+C_6\,\mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right]+\left(C\cdot C_4+2\left(p-2\right)C^{\frac{p}{2}}\right)T\right)\\&\leqslant\left(1+\frac{1}{C_7}\right)\left(1+C_6\right)\left(1+\left(C\cdot C_4+2\left(p-2\right)C^{\frac{p}{2}}\right)T\right)e^{C_5T}\left(\mathbb{E}\left[\|X_0\|_H^p\right]+\mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right]+1\right),\end{aligned}\tag{2.3.13}$$

where the constant

$$C_2:=C_2\left(p,\gamma,\theta,C,K,T\right):=\left(1+\frac{1}{C_7}\right)\left(1+C_6\right)\left(1+\left(C\cdot C_4+2\left(p-2\right)C^{\frac{p}{2}}\right)T\right)e^{C_5T}$$

is independent of R, t, n and the stopping time $\tau_R^{(n)}$.
Step (ii).8. In this last step we apply the monotone convergence theorem. Since (2.3.13) holds for $T\in[0,T]$ and $\tau_R^{(n)}\to T$ as $R\to\infty$ P-a.s. we get

$$\begin{aligned}&\sup_{r\in[0,T]}\mathbb{E}\left[\left\|X_r^{(n)}\right\|_H^p\right]+\mathbb{E}\left[\int_0^T\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right]\\&=\lim_{R\to\infty}\sup_{r\in\left[0,T\wedge\tau_R^{(n)}\right]}\mathbb{E}\left[\left\|X_r^{(n)}\right\|_H^p\right]+\mathbb{E}\left[\lim_{R\to\infty}\int_0^{T\wedge\tau_R^{(n)}}\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right]\\&=\lim_{R\to\infty}\left(\sup_{r\in\left[0,T\wedge\tau_R^{(n)}\right]}\mathbb{E}\left[\left\|X_r^{(n)}\right\|_H^p\right]+\mathbb{E}\left[\int_0^{T\wedge\tau_R^{(n)}}\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right]\right)\\&\overset{(2.3.13)}{\leqslant}C_2\left(\mathbb{E}\left[\|X_0\|_H^p\right]+\mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right]+1\right)\end{aligned}$$

for all $n\in\mathbb{N}$.
Now let us prove 2.3.5 (iii).
Step (iii).1. Again, we apply Itô's formula 2.3.3 to the process $X_t^{(n)}$ (see (2.3.11)). Next, we use the results from Step (ii).2 and (ii).3 to calculate the occuring terms $I_1\left(\hat{t}\right)$ and

$I_2\left(\hat{t}\right)$ in the stopped version of (2.3.11). Then we apply the absolute value and the triangle inequality to get

$$\begin{aligned}
&\left\|X_{\hat{t}}^{(n)}\right\|_H^p+\frac{p\theta}{2}\int_0^{\hat{t}}\left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\|X_{s-}^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\\
\leqslant\;&\left\|X_0^{(n)}\right\|_H^2+\left(p\left(p-2\right)\gamma\right)\int_0^{\hat{t}}\left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\|X_{s-}^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\\
&+\left(p\left(p-2\right)\frac{2}{p}C^{\frac{p}{2}}+1\right)\int_0^{\hat{t}}F_s^{\frac{p}{2}}\,\mathrm{d}s\\
&+\left(p\left(p-2\right)\left(C+\frac{2\left(p-2\right)}{p}\right)+\frac{p}{2}\left(K+\frac{p-2}{p}\right)\right)\int_0^{\hat{t}}\left\|X_{s-}^{(n)}\right\|_H^p\,\mathrm{d}s+\left(p-2\right)2TC^{\frac{p}{2}}\\
&+p\left|\int_0^{\hat{t}}\left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\langle X_{s-}^{(n)},P_nB\left(s,X_s^{(n)}\right)\mathrm{d}W_s^{(n)}\right\rangle_H\right|\\
&+p\left|\int_0^{\hat{t}}\int_Z\left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\langle X_{s-}^{(n)},P_nf\left(s,X_{s-}^{(n)},z\right)\right\rangle_H\bar{\mu}\left(\mathrm{d}s,\mathrm{d}z\right)\right|\\
&+\left|\int_0^{\hat{t}}\int_Z\left(\left\|X_{s-}^{(n)}+P_nf\left(s,X_{s-}^{(n)},z\right)\right\|_H^p-\left\|X_{s-}^{(n)}\right\|_H^p\right.\right.\\
&\qquad\left.\left.-p\left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\langle X_{s-}^{(n)},P_nf\left(s,X_s^{(n)},z\right)\right\rangle_H\right)\mu\left(\mathrm{d}s,\mathrm{d}z\right)\right|.
\end{aligned}\tag{2.3.14}$$

for $t\in[0,T]$. On (2.3.14) we apply the supremum over $\left[0,\tau_R^{(n)}\wedge t\right]=\left[0,\hat{t}\right]$. The Lebesgue-integrals stay unchanged, because all integrands are non-negative (also F was chosen to be non-negative) and hence the integrals are increasing in time. So we have

$$\begin{aligned}
\sup_{r\in\left[0,\hat{t}\right]}\left\|X_r^{(n)}\right\|_H^p+&\quad\frac{p\theta}{2}\int_0^{\hat{t}}\left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\|X_{s-}^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\\
&\leqslant\;\left\|X_0^{(n)}\right\|_H^2+C_8\int_0^{\hat{t}}\left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\|X_{s-}^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s+C_9\int_0^{\hat{t}}F_s^{\frac{p}{2}}\,\mathrm{d}s\\
&\quad+C_{10}\int_0^{\hat{t}}\left\|X_{s-}^{(n)}\right\|_H^p\,\mathrm{d}s+pJ_1\left(\hat{t}\right)+pJ_2\left(\hat{t}\right)+J_3\left(\hat{t}\right)+\left(p-2\right)2TC^{\frac{p}{2}},
\end{aligned}\tag{2.3.15}$$

with $C_8{=}p\left(p-2\right)\gamma,\ C_9{=}p\left(p-2\right)\frac{2}{p}C^{\frac{p}{2}}+1,\ C_{10}{=}p\left(p-2\right)\left(C+\frac{2(p-2)}{p}\right)+\frac{p}{2}\left(K+\frac{p-2}{p}\right)$

and

$$J_1\left(\hat{t}\right) = \sup_{r\in\left[0,\hat{t}\right]} \left| \int_0^r \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\langle X_{s-}^{(n)}, P_n B\left(s, X_s^{(n)}\right) \mathrm{d}W_s^{(n)}\right\rangle_H \right|,$$

$$J_2\left(\hat{t}\right) = \sup_{r\in\left[0,\hat{t}\right]} \left| \int_0^r \int_Z \left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\langle X_{s-}^{(n)}, P_n f\left(s, X_{s-}^{(n)}, z\right)\right\rangle_H \bar{\mu}\left(\mathrm{d}s, \mathrm{d}z\right) \right|,$$

$$J_3\left(\hat{t}\right) = \sup_{r\in\left[0,\hat{t}\right]} \left| \int_0^r \int_Z \left(\left\|X_{s-}^{(n)} + P_n f\left(s, X_{s-}^{(n)}, z\right)\right\|_H^p - \left\|X_{s-}^{(n)}\right\|_H^p \right.\right.$$
$$\left.\left. -p\left\|X_{s-}^{(n)}\right\|_H^{p-2} \left\langle X_{s-}^{(n)}, P_n f\left(s, X_s^{(n)}, z\right)\right\rangle_H \right) \mu\left(\mathrm{d}s, \mathrm{d}z\right) \right|.$$

We want to estimate J_1, J_2 and J_3 from above by the Lebesgue-integrals, which already appeared, in expectation in the next three steps.

Step (iii).2. Let us estimate $\mathbb{E}\left[J_1\left(\hat{t}\right)\right]$. Since J_1 without the supremum and absolute value is a real-valued, local martingale by Step (ii).5, we may apply the Burkholder-Davis-Gundy inequality D.5 (i) and then condition (B1) in the form of (2.3.8). Remember that $C_{BDG} > 0$ is the generic constant from D.5 (i). Then

$$\begin{aligned}
\mathbb{E}\left[J_1\left(\hat{t}\right)\right] &= \mathbb{E}\left[\sup_{r\in\left[0,\hat{t}\right]} \left| \int_0^r \left\|X_s^{(n)}\right\|_H^{p-2} \left\langle X_s^{(n)}, P_n B\left(s, X_s^{(n)}\right) \mathrm{d}W_s^{(n)}\right\rangle_H \right|\right] \\
&\overset{D.5\,(i)}{\leqslant} C_{BDG}\,\mathbb{E}\left[\left(\int_0^{\hat{t}} \left\|X_s^{(n)}\right\|_H^{2p-2} \left\|B\left(s, X_s^{(n)}\right)\right\|_{L_2}^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leqslant C_{BDG}\,\mathbb{E}\left[\left(\sup_{r\in\left[0,\hat{t}\right]} \left\|X_r^{(n)}\right\|_H^p \int_0^{\hat{t}} \left\|X_s^{(n)}\right\|_H^{p-2} \left\|B\left(s, X_s^{(n)}\right) \mathrm{d}s\right\|_{L_2}^2\right)^{\frac{1}{2}}\right] \\
&\overset{(2.3.8)}{\leqslant} C_{BDG}\,\mathbb{E}\left[\left(\sup_{r\in\left[0,\hat{t}\right]} \left\|X_r^{(n)}\right\|_H^p \int_0^{\hat{t}} \left\|X_s^{(n)}\right\|_H^{p-2} \left(C\left(1 + F_s + \|X_s^{(n)}\|_H^2\right) + \gamma\|X_s^{(n)}\|_V^\alpha\right) \mathrm{d}s\right)^{\frac{1}{2}}\right].
\end{aligned}$$

Let $\varepsilon > 0$ arbitrary. We want to apply Young's inequality with $q = 2 = q'$ to the right hand side of the equation above which is equal to

$$
\begin{aligned}
&\mathbb{E}\left[\sqrt{2\varepsilon}\left(\sup_{r\in[0,\hat{t}]}\left\|X_r^{(n)}\right\|_H^p\right)^{\frac{1}{2}}\right.\\
&\qquad\left.\cdot\frac{C_{BDG}}{\sqrt{2\varepsilon}}\left(\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}\left(C\left(1+F_s+\|X_s^{(n)}\|_H^2\right)+\gamma\|X_s^{(n)}\|_V^\alpha\right)\mathrm{d}s\right)^{\frac{1}{2}}\right]\\
\overset{Young}{\leqslant}&\,\varepsilon\,\mathbb{E}\left[\sup_{r\in[0,\hat{t}]}\left\|X_r^{(n)}\right\|_H^p\right]+C_{11}\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}\left(C\left(1+F_s+\|X_s^{(n)}\|_H^2\right)+\gamma\|X_s^{(n)}\|_V^\alpha\right)\mathrm{d}s\right]\\
=&\varepsilon\,\mathbb{E}\left[\sup_{r\in[0,\hat{t}]}\left\|X_r^{(n)}\right\|_H^p\right]+C\cdot C_{11}\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^p\mathrm{d}s\right]+C\cdot C_{11}\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}F_s\,\mathrm{d}s\right]\\
&+\gamma C_{11}\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^\alpha\mathrm{d}s\right]+C\cdot C_{11}\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}\mathrm{d}s\right]
\end{aligned}
\tag{2.3.16}
$$

where $C_{11}=C_{11}(\varepsilon,C_{BDG})=\frac{C_{BDG}^2}{4\varepsilon}$. Again with Young's inequality applied with $q=\frac{p}{p-2}$, $q'=\frac{p}{2}$ on the $\left\|X_s^{(n)}\right\|_H^{p-2}$-terms we arrive at

$$
\begin{aligned}
\mathbb{E}\left[J_1\left(\hat{t}\right)\right]\leqslant\;&\varepsilon\,\mathbb{E}\left[\sup_{r\in[0,\hat{t}]}\left\|X_r^{(n)}\right\|_H^p\right]+C_{11}\cdot C\left(1+\frac{2(p-2)}{p}\right)\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^p\mathrm{d}s\right]\\
&+\frac{2C\cdot C_{11}}{p}\,\mathbb{E}\left[\int_0^{\hat{t}}F_s\,\mathrm{d}s\right]+\gamma C_{11}\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^\alpha\mathrm{d}s\right]+C\cdot C_{11}T
\end{aligned}
$$

for all $t\in[0,T]$.

Step (iii).3. Now we come to $\mathbb{E}\left[J_2\left(\hat{t}\right)\right]$. By Step (ii).4, J_2 without the supremum and absolute value is a martingale and again, we may apply the Burkholder-Davis-Gundy inequality D.5 (i) with the already given, generic constant $C_{BDG}>0$, cf. Step (iii).2.

Then condition (B1) in form of (2.3.12) gives us, for an arbitrary $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{E}\left[J_2\left(\hat{t}\right)\right] &= \mathbb{E}\left[\sup_{r\in\left[0,\hat{t}\right]}\left|\int_0^r\int_Z \left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\langle X_{s-}^{(n)}, P_n f\left(s, X_{s-}^{(n)}, z\right)\right\rangle_H \bar{\mu}\left(\mathrm{d}s,\mathrm{d}z\right)\right|\right] \\
&\overset{D.5\,(i)}{\leqslant} C_{BDG}\,\mathbb{E}\left[\left(\int_0^{\hat{t}}\int_Z \left\|X_s^{(n)}\right\|_H^{2p-2}\left\|f\left(s, X_s^{(n)}, z\right)\right\|_H^2 m\left(\mathrm{d}z\right)\,\mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leqslant C_{BDG}\,\mathbb{E}\left[\left(\sup_{r\in\left[0,\hat{t}\right]}\left\|X_r^{(n)}\right\|_H^p\int_0^{\hat{t}}\int_Z \left\|X_s^{(n)}\right\|_H^{p-2}\left\|f\left(s, X_s^{(n)}, z\right)\right\|_H^2 m\left(\mathrm{d}z\right)\,\mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\overset{(2.3.12)}{\leqslant} C_{BDG}\,\mathbb{E}\left[\left(\sup_{r\in\left[0,\hat{t}\right]}\left\|X_r^{(n)}\right\|_H^p\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}\left(C\left(1+F_s+\left\|X_s^{(n)}\right\|_H^2\right)+\gamma\left\|X_s^{(n)}\right\|_V^\alpha\right)\mathrm{d}s\right)^{\frac{1}{2}}\right]. \\
&= \mathbb{E}\left[\sqrt{2\varepsilon}\left(\sup_{r\in\left[0,\hat{t}\right]}\left\|X_r^{(n)}\right\|_H^p\right)^{\frac{1}{2}}\right. \\
&\qquad\left.\cdot\frac{C_{BDG}}{\sqrt{2\varepsilon}}\left(\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}\left(C\left(1+F_s+\left\|X_s^{(n)}\right\|_H^2\right)+\gamma\left\|X_s^{(n)}\right\|_V^\alpha\right)\mathrm{d}s\right)^{\frac{1}{2}}\right].
\end{aligned}$$

Since this is exactly the same situation as in (2.3.16), we directly conclude

$$\begin{aligned}
\mathbb{E}\left[J_2\left(\hat{t}\right)\right] = {}& \varepsilon\,\mathbb{E}\left[\sup_{r\in\left[0,\hat{t}\right]}\left\|X_r^{(n)}\right\|_H^p\right] + C\cdot C_{11}\left(1+\frac{2\left(p-2\right)}{p}\right)\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^p\,\mathrm{d}s\right] \\
&+\frac{2C\cdot C_{11}}{p}\,\mathbb{E}\left[\int_0^{\hat{t}}F_s\,\mathrm{d}s\right]+\gamma C_{11}\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^\alpha\,\mathrm{d}s\right]+C\cdot C_{11}T
\end{aligned}$$

for all $t \in [0,T]$.

Step (iii).4. For the term $\mathbb{E}\left[J_3\left(\hat{t}\right)\right]$ we have by Proposition 1.2.7 and Lemma C.1

$$\begin{aligned}\mathbb{E}\left[J_3\left(\hat{t}\right)\right] \leqslant\ & \mathbb{E}\Bigg[\int_0^{\hat{t}}\int_Z\Bigg|\left\|X_{s-}^{(n)}+P_nf\left(s,X_{s-}^{(n)},z\right)\right\|_H^p-\left\|X_{s-}^{(n)}\right\|_H^p \\ &\qquad -p\left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\langle X_{s-}^{(n)},P_nf\left(s,X_s^{(n)},z\right)\right\rangle_H\Bigg|\,\mu\left(\mathrm{d}s,\mathrm{d}z\right)\Bigg] \\ \overset{1.2.7}{=}\ & \mathbb{E}\Bigg[\int_0^{\hat{t}}\int_Z\Bigg|\left\|X_{s}^{(n)}+P_nf\left(s,X_{s}^{(n)},z\right)\right\|_H^p-\left\|X_{s}^{(n)}\right\|_H^p \\ &\qquad -p\left\|X_{s}^{(n)}\right\|_H^{p-2}\left\langle X_{s}^{(n)},P_nf\left(s,X_s^{(n)},z\right)\right\rangle_H\Bigg|\,m\left(\mathrm{d}z\right)\,\mathrm{d}s\Bigg] \\ \overset{C.1}{\leqslant}\ & C_{12}\,\mathbb{E}\left[\int_0^{\hat{t}}\int_Z\left(\left\|X_s^{(n)}\right\|_H^p+\left\|f\left(s,X_s,z\right)\right\|_H^p\right)m\left(\mathrm{d}z\right)\,\mathrm{d}s\right]\end{aligned}$$

with $C_{12}=C_{12}\left(p\right)=p+2^{p-1}\left(2p-3\right)$ (cf. proof of Lemma C.1). Now we apply (B2) to the right hand side and arrive at

$$\begin{aligned}\mathbb{E}\left[J_3\left(\hat{t}\right)\right] \leqslant & \left(C_{12}+C\right)\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^p\,\mathrm{d}s\right]+C\cdot C_{12}\,\mathbb{E}\left[\int_0^{\hat{t}}F_s^{\frac{p}{2}}\,\mathrm{d}s\right] \\ & +\gamma C_{12}\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right]+C\cdot C_{12}T\end{aligned}$$

for all $t\in[0,T]$.

Step (iii).5. We want to apply Gronwall's inequality. But first, let us combine the results from Steps (iii).2 to (iii).4 and apply them to (2.3.15) in expectation. Then for all $t\in[0,T]$ and $\varepsilon>0$

$$\begin{aligned}&\mathbb{E}\left[\sup_{r\in\left[0,\hat{t}\right]}\left\|X_r^{(n)}\right\|_H^p\right]+\frac{p\theta}{2}\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right] \\ \leqslant\ & \mathbb{E}\left[\left\|X_0\right\|_H^p\right]+\left(C_9+4C\cdot C_{11}+C\cdot C_{12}\right)\mathbb{E}\left[\int_0^{\hat{t}}F_s^{\frac{p}{2}}\,\mathrm{d}s\right]+T\left(C\cdot C_{12}+2C\cdot C_{11}+2\left(p-2\right)C^{\frac{p}{2}}\right) \\ & +\left(C_{10}+2p\cdot C_{11}\cdot C\left(1+\frac{2\left(p-2\right)}{p}\right)+C_{12}+C\right)\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^p\,\mathrm{d}s\right] \\ & +\left(C_8+2p\gamma C_{11}+\gamma C_{12}\right)\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right]+2\varepsilon p\,\mathbb{E}\left[\sup_{r\in\left[0,\hat{t}\right]}\left\|X_r^{(n)}\right\|_H^p\right].\end{aligned} \tag{2.3.17}$$

We used $\left\|X_0^{(n)}\right\|_H \leqslant \|X_0\|_H$ here. We choose $\varepsilon = \frac{1}{3p}$ and define

$$\begin{aligned} C_{13} := C_{13}\left(p,\theta,\gamma,C_{BDG}\right) &:= \frac{p\theta}{2} - \left(C_8 + 2p\gamma C_{11} + \gamma C_{12}\right) \\ &= \frac{p\theta}{2} - \gamma\left(p\left(p-2\right) + \frac{2pC_{BDG}^2}{4\varepsilon} + p + 2^{p-1}\left(2p-3\right)\right) \\ &= \frac{p\theta}{2} - \gamma\left(p\left(p-2\right) + \frac{3}{2}p^2C_{BDG}^2 + p + 2^{p-1}\left(2p-3\right)\right). \end{aligned}$$

Then $C_{13} > 0$, since we assumed $\gamma < \Gamma$. Furthermore we set

$$C_{14} := C_{14}\left(C_{BDG},p\right) := \left(2 + 3C_{BDG}^2\right)p - 3 + C_{12},$$

$$C_{15} := C_{15}\left(p,K,C,C_{BDG}\right) := C\left(1 + \frac{3}{2}p^2C_{BDG}^2\left(1 + \frac{2\left(p-2\right)}{p}\right)\right) + C_{10} + C_{12} \quad \text{and}$$

$$C_{16} := C_{16}\left(p,T,C,C_{BDG}\right) := T\left(C\cdot C_{12} + \frac{3}{2}p\cdot C\cdot C_{BDG}^2 + 2\left(p-2\right)C^{\frac{p}{2}}\right).$$

Now bringing the last two summands to the left hand side in (2.3.17) yields to

$$\begin{aligned} &\frac{1}{3}\,\mathbb{E}\left[\sup_{r\in[0,\hat{t}]}\left\|X_r^{(n)}\right\|_H^p\right] + C_{13}\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_{s-}^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right] \\ &\qquad\leqslant \mathbb{E}\left[\|X_0\|_H^p\right] + C_{14}\,\mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right] + C_{15}\,\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^p\,\mathrm{d}s\right] + C_{16}. \end{aligned} \tag{2.3.18}$$

By the definition of our stopping time $\tau_R^{(n)}$ the right hand side is finite. Since $0 \leqslant \left\|X_s^{(n)}\right\|_H^p \leqslant \sup_{r\in[0,s]}\left\|X_r^{(n)}\right\|_H^p$ for all $s \in [0,T]$ and since the integrals are isotone we have by Fubini's theorem

$$\mathbb{E}\left[\int_0^{\hat{t}}\left\|X_s^{(n)}\right\|_H^p\,\mathrm{d}s\right] = \int_0^{\hat{t}}\mathbb{E}\left[\left\|X_s^{(n)}\right\|_H^p\right]\mathrm{d}s \leqslant \int_0^{\hat{t}}\mathbb{E}\left[\sup_{r\in[0,s]}\left\|X_r^{(n)}\right\|_H^p\right]\mathrm{d}s.$$

For simplicity we define

$$\begin{aligned} \varphi\left(t\right) &:= \mathbb{E}\left[\sup_{r\in[0,t]}\left\|X_r^{(n)}\right\|_H^p\right] \\ \psi\left(t\right) &:= \mathbb{E}\left[\int_0^t\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right]. \end{aligned}$$

Then we can rewrite (2.3.18) in the following way:

$$\varphi\left(\hat{t}\right) + 3C_{13}\psi\left(\hat{t}\right) \leqslant 3\Theta + \int_0^{\hat{t}} 3C_{15}\varphi\left(s\right)\,\mathrm{d}s$$

for all $t \in [0,T]$, where $\Theta := \mathbb{E}\left[\|X_0\|_H^p\right] + C_{14}\,\mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right] + C_{16}$. Note that $C_{15} > 0$ because $K > 0$. Since ψ is non-negative we have

$$\varphi\left(\hat{t}\right) + 3C_{13}\psi\left(\hat{t}\right) \leqslant 3\Theta + \int_0^{\hat{t}} 3C_{15}\varphi\left(s\right)\,\mathrm{d}s \leqslant 3\Theta + \int_0^{\hat{t}} 3C_{15}\left(\varphi\left(s\right) + 3C_{13}\psi\left(s\right)\right)\,\mathrm{d}s$$

and we can apply Gronwall's inequality B.5 to get

$$\varphi\left(\hat{t}\right) + 3C_{13}\psi\left(\hat{t}\right) \leqslant 3\Theta e^{3C_{15}\hat{t}} \leqslant 3\Theta e^{3C_{15}T}.$$

The non-negativity of φ and ψ and the fact that $C_{13} > 0$ gives us $\varphi\left(\hat{t}\right) \leqslant \varphi\left(\hat{t}\right) + 3C_{13}\psi\left(\hat{t}\right) \leqslant 3\Theta e^{3C_{15}T}$ and

$$\psi\left(\hat{t}\right) \leqslant \frac{1}{3C_{13}}\left(\varphi\left(\hat{t}\right) + 3C_{13}\psi\left(\hat{t}\right)\right) \leqslant \frac{1}{C_{13}}\Theta e^{3C_{15}T}.$$

Hence we have for all $t \in [0,T]$ and $n \in \mathbb{N}$

$$\begin{aligned}\varphi\left(\hat{t}\right) + \psi\left(\hat{t}\right) &\leqslant \left(3 + \frac{1}{C_{13}}\right)e^{3C_{15}T}\left(\mathbb{E}\left[\|X_0\|_H^p\right] + C_{14}\,\mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right] + C_{16}\right)\\ &\leqslant \left(3 + \frac{1}{C_{13}}\right)\left(1 + C_{14}\right)\left(1 + C_{16}\right)e^{3C_{15}T}\left(\mathbb{E}\left[\|X_0\|_H^p\right] + \mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right] + 1\right),\end{aligned} \tag{2.3.19}$$

where the constant $C_3 := C_3\left(p,\gamma,\theta,C,C_{BDG},K,T\right) := \left(3 + \frac{1}{C_{13}}\right)\left(1 + C_{14}\right)\left(1 + C_{16}\right)e^{3C_{15}T}$ is independent of R, t, n and the stopping time $\tau_R^{(n)}$.

Step (iii).6. Finally we apply the monotone convergence theorem. Since (2.3.19) holds for $T \in [0,T]$ and $\tau_R^{(n)} \to T$ as $R \to \infty$ P-a.s. we get

$$\begin{aligned}&\mathbb{E}\left[\sup_{r\in[0,T]}\left\|X_r^{(n)}\right\|_H^p\right] + \mathbb{E}\left[\int_0^T \left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right]\\ =&\,\mathbb{E}\left[\lim_{R\to\infty}\left(\sup_{r\in\left[0,T\wedge\tau_R^{(n)}\right]}\left\|X_r^{(n)}\right\|_H^p + \int_0^{T\wedge\tau_R^{(n)}}\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_s^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right)\right]\\ =&\,\lim_{R\to\infty}\left(\mathbb{E}\left[\sup_{r\in\left[0,T\wedge\tau_R^{(n)}\right]}\left\|X_r^{(n)}\right\|_H^p\right] + \mathbb{E}\left[\int_0^{T\wedge\tau_R^{(n)}}\left\|X_s^{(n)}\right\|_H^{p-2}\left\|X_{s-}^{(n)}\right\|_V^{\alpha}\,\mathrm{d}s\right]\right)\\ \overset{(2.3.19)}{\leqslant}&\,C_3\left(\mathbb{E}\left[\|X_0\|_H^p\right] + \mathbb{E}\left[\int_0^T F_s^{\frac{p}{2}}\,\mathrm{d}s\right] + 1\right)\end{aligned}$$

for all $n \in \mathbb{N}$.

□

Notation. *To simplify the used spaces in the following, we introduce the abbreviations*

$$\begin{aligned}\mathfrak{L}^{\alpha} &= L^{\alpha}\left([0,T]\times\Omega, \mathrm{d}t\otimes P; V\right),\\ \mathfrak{L}^{\alpha'} &= L^{\frac{\alpha}{\alpha-1}}\left([0,T]\times\Omega, \mathrm{d}t\otimes P; V^{*}\right),\\ \mathfrak{L}^{2} &= L^{2}\left([0,T]\times\Omega, \mathrm{d}t\otimes P, L_2\left(U;H\right)\right),\\ \mathfrak{M} &= \mathcal{M}_T^2\left(\mathcal{P}\otimes\mathcal{Z}, \mathrm{d}t\otimes P\otimes m; H\right).\end{aligned}$$

2.3.7 Lemma. *Suppose conditions (A1)–(A4) and (B1)–(B3) hold and that $F \in L^{\frac{p}{2}}([0,T]\times\Omega, \mathrm{d}t\times P)$. For each $n\in\mathbb{N}$ let $\left(X_t^{(n)}\right)_{t\in[0,T]}$ be a solution to (2.3.1). Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and elements $\bar{X}\in\mathfrak{L}^{\alpha}\cap L^{\infty}\left([0,T];L^p\left(\Omega;H\right)\right)$, $Y\in\mathfrak{L}^{\alpha'}$, $Z\in\mathfrak{L}^2$, $g\in\mathfrak{M}$ such that the following holds:*

(i) $X^{(n_k)}\to\bar{X}$ *weakly in* $\mathfrak{L}^{\alpha}$ *and weakly star in* $L^{\infty}\left([0,T];L^p\left(\Omega;H\right)\right)$ *as* $k\to\infty$.

(ii) $P_{n_k}A\left(\cdot, X^{(n_k)}\right)\to Y$ *weakly in* $\mathfrak{L}^{\alpha'}$ *as* $k\to\infty$.

(iii) $P_{n_k}B\left(\cdot, X^{(n_k)}\right)\to Z$ *weakly in* $\mathfrak{L}^2$ *and*

$$\int_0^{\cdot} P_{n_k}B\left(s, X_s^{(n_k)}\right)\mathrm{d}W_s^{(n_k)} \to \int_0^{\cdot} Z_s\,\mathrm{d}W_s$$

weakly in $L^{\infty}\left([0,T];L^2\left(\Omega;H\right)\right)$ *as* $k\to\infty$.

(iv) $P_{n_k}f\left(\cdot, X^{(n_k)},\cdot\right)\to g$ *weakly in* $\mathfrak{M}$ *and*

$$\int_0^{\cdot}\int_Z P_{n_k}f\left(s, X_s^{(n_k)}, z\right)\bar{\mu}\left(\mathrm{d}s,\mathrm{d}z\right)\to\int_0^{\cdot} g\left(s,z\right)\bar{\mu}\left(\mathrm{d}s,\mathrm{d}z\right)$$

weakly in $L^{\infty}\left([0,T];L^2\left(\Omega;H\right)\right)$ *as* $k\to\infty$.

Proof. **Part (i)**. By Lemma 2.3.5 (i) we know that

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\int_0^T\left\|X_t^{(n)}\right\|_V^{\alpha}\mathrm{d}t\right]<\infty,$$

i.e. the sequence $\left(X^{(n)}\right)_{n\in\mathbb{N}}$ is bounded in $\mathfrak{L}^{\alpha}$. Since $1<\alpha<\infty$, $\mathfrak{L}^{\alpha}$ is reflexive and hence there exists a weakly convergent subsequence $\left(X^{(n_k)}\right)_{k\in\mathbb{N}}$ and an element $\bar{X}\in\mathfrak{L}^{\alpha}$ such that $X^{(n_k)}\to\bar{X}$ weakly as $k\to\infty$.
Furthermore Lemma 2.3.5 (ii) tells us that

$$\sup_{k\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\left[\left\|X_t^{(n_k)}\right\|_H^p\right]<\infty.$$

So the sequence $\left(X^{(n_k)}\right)_{k\in\mathbb{N}}$ is bounded in $L^{\infty}\left([0,T];L^p\left(\Omega;H\right)\right)$. We can identify $L^{\infty}([0,T];L^p\left(\Omega;H\right)) = \left(L^1\left([0,T];L^{\frac{p}{p-1}}\left(\Omega;H\right)\right)\right)^{*}$ and by the Banach-Alaoglu theorem

E.1 there exists another weakly star convergent subsequence $\left(X^{(n'_k)}\right)_{k\in\mathbb{N}}$ and an element $\hat{X} \in L^\infty([0,T]; L^p(\Omega; H))$ such that $X^{(n'_k)} \to \hat{X}$ weakly star as $k \to \infty$. But we also have $X^{(n'_k)} \to \bar{X}$ weakly as $k \to \infty$, so we conclude $\bar{X} = \hat{X}$.

Part (ii). Also the space $\mathfrak{L}^{\alpha'}$ is reflexive (since $1 < \alpha < \infty$) and so we only have to show that the sequence $\left(P_{n'_k} A\left(\cdot, X^{(n'_k)}\right)\right)_{k\in\mathbb{N}}$, where $(n'_k)_{k\in\mathbb{N}}$ is the subsequence from the last step, is bounded in $\mathfrak{L}^{\alpha'}$. Then there exists another subsequence $(n''_k)_{k\in\mathbb{N}}$, and an element $Y \in \mathfrak{L}^{\alpha'}$ such that $P_{n'_k} A\left(\cdot, X^{(n''_k)}\right) \to Y$ weakly as $k \to \infty$. We have by (A4) and Young's inequality (remember $p = \beta + 2$)

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \mathbb{E}\left[\int_0^T \left\| A\left(t, X_t^{(n'_k)}\right)\right\|_{V^*}^{\frac{\alpha}{\alpha-1}} \mathrm{d}t\right] \\
&\overset{(A4)}{\leqslant} \sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T \left(F_t + K\left\|X_t^{(n'_k)}\right\|_V^\alpha\right)\left(1 + \left\|X_t^{(n'_k)}\right\|_H^\beta\right) \mathrm{d}t\right] \\
&= \sup_{k\in\mathbb{N}} \mathbb{E}\left[\int_0^T \left(F_t + K\left\|X_t^{(n'_k)}\right\|_V^\alpha + F_t\left\|X_t^{(n'_k)}\right\|_H^{p-2} + K\left\|X_t^{(n'_k)}\right\|_V^\alpha \left\|X_t^{(n'_k)}\right\|_H^\beta\right) \mathrm{d}t\right] \\
&\overset{Young}{\leqslant} \sup_{k\in\mathbb{N}} \mathbb{E}\left[\int_0^T \left(F_t + K\left\|X_t^{(n'_k)}\right\|_V^\alpha + \frac{2}{p} F_t^{\frac{p}{2}} + \frac{p-2}{p}\left\|X_t^{(n'_k)}\right\|_H^p \right.\right. \\
&\qquad\qquad \left.\left. + K\left\|X_t^{(n'_k)}\right\|_V^\alpha \left\|X_t^{(n'_k)}\right\|_H^\beta\right) \mathrm{d}t\right].
\end{aligned} \tag{2.3.20}$$

We use Lemma 2.3.5 (i) to get

$$K \sup_{k\in\mathbb{N}} \mathbb{E}\left[\int_0^T \left\|X_t^{(n'_k)}\right\|_V^\alpha \mathrm{d}t\right] < \infty.$$

Lemma 2.3.5 (ii) gives us

$$K \sup_{k\in\mathbb{N}} \mathbb{E}\left[\int_0^T \left\|X_t^{(n'_k)}\right\|_V^\alpha \left\|X_t^{(n'_k)}\right\|_H^\beta \mathrm{d}t\right] < \infty$$

and with Fubini's theorem

$$\begin{aligned}
\frac{p-2}{p} \sup_{k\in\mathbb{N}} \mathbb{E}\left[\int_0^T \left\|X_t^{(n'_k)}\right\|_H^p \mathrm{d}t\right] &= \frac{p-2}{p} \sup_{k\in\mathbb{N}} \int_0^T \mathbb{E}\left[\left\|X_t^{(n'_k)}\right\|_H^p\right] \mathrm{d}t \\
&\leqslant \frac{p-2}{p} \sup_{k\in\mathbb{N}} \int_0^T \sup_{s\in[0,T]} \mathbb{E}\left[\left\|X_s^{(n'_k)}\right\|_H^p\right] \mathrm{d}t \\
&= \frac{T(p-2)}{p} \sup_{t\in[0,T]} \mathbb{E}\left[\left\|X_t^{(n'_k)}\right\|_H^p\right] \overset{(2.3.4)}{<} \infty.
\end{aligned}$$

Finally, by the assumption that $F \in L^{\frac{p}{2}}([0,T]\times\Omega, \mathrm{d}t\times P)$ and Young's inequality, we see that

$$\mathbb{E}\left[\int_0^T \left(F_t + \frac{2}{p}F_t^{\frac{p}{2}}\right)\mathrm{d}t\right] \leqslant \mathbb{E}\left[\int_0^T \left(\frac{4}{p}F_t^{\frac{p}{2}} + \frac{p-2}{p}\right)\mathrm{d}t\right]$$
$$= \frac{4}{p}\,\mathbb{E}\left[\int_0^T F_t^{\frac{p}{2}}\,\mathrm{d}t\right] + \frac{T\,(p-2)}{p} < \infty,$$

hence (2.3.20) is finite and so there exists the required subsequence $(n''_k)_{k\in\mathbb{N}} \subset (n'_k)_{k\in\mathbb{N}}$.
Part (iii). As before, since the space $\mathfrak{L}^2$ is reflexive, too, it is sufficient to show that $\left(P_{n''_k}B\left(\cdot, X^{(n''_k)}\right)\right)_{k\in\mathbb{N}}$ is bounded in $\mathfrak{L}^2$. Condition (B1), Fubini's theorem and Lemma 2.3.5 (i) give us

$$\begin{aligned}
&\sup_{k\in\mathbb{N}}\mathbb{E}\left[\int_0^T \left\|P_{n''_k}B\left(t, X_t^{(n''_k)}\right)\right\|_{L_2}^2 \mathrm{d}t\right]\\
&\overset{\text{(B1)}}{\leqslant} \sup_{k\in\mathbb{N}}\mathbb{E}\left[\int_0^T \left(C\left(1+F_t+\left\|X_t^{(n''_k)}\right\|_H^2\right)+\gamma\left\|X_t^{(n''_k)}\right\|_V^\alpha\right)\mathrm{d}t\right]\\
&=\sup_{k\in\mathbb{N}}\left(CT + C\,\mathbb{E}\left[\int_0^T F_t\,\mathrm{d}t\right] + C\int_0^T \mathbb{E}\left[\left\|X_t^{(n''_k)}\right\|_H^2\right]\mathrm{d}t + \gamma\,\mathbb{E}\left[\int_0^T \left\|X_t^{(n''_k)}\right\|_V^\alpha \mathrm{d}t\right]\right)\\
&\overset{(2.3.3)}{\leqslant} \sup_{k\in\mathbb{N}}\left(CT + C\,\mathbb{E}\left[\int_0^T F_t\,\mathrm{d}t\right] + CT\sup_{t\in[0,T]}\mathbb{E}\left[\left\|X_t^{(n''_k)}\right\|_H^2\right]\right) + \gamma C_1\\
&\overset{(2.3.3)}{\leqslant} CT + C\,\mathbb{E}\left[\int_0^T F_t\,\mathrm{d}t\right] + C_1\,(CT+\gamma) < \infty,
\end{aligned} \tag{2.3.21}$$

since by Hölder's inequality $\mathbb{E}\left[\int_0^T F_t\,\mathrm{d}t\right] \leqslant T^{\frac{p-2}{p}}\,\mathbb{E}\left[\int_0^T F_t^{\frac{p}{2}}\,\mathrm{d}t\right]^{\frac{2}{p}} < \infty$. So there exists a subsequence $(n'''_k)_{k\in\mathbb{N}}$ such that $\left(P_{n'''_k}B\left(\cdot, X^{(n'''_k)}\right)\right)_{k\in\mathbb{N}}$ converges weakly in $\mathfrak{L}^2$ to an element $Z \in \mathfrak{L}^2$.
Let us come to the second part of (iii): Since $\tilde{P}_n$ is the orthogonal projection onto $\mathrm{Span}\{g_1,\ldots,g_n\}$ in U, without loss of generality we have that $P_{n'''_k}B\left(t, X_t^{(n'''_k)}\right)\tilde{P}_{n'''_k}$ converges weakly to Z in $\mathfrak{L}^2$. Furthermore

$$\int_0^\cdot P_nB\left(s, X_t^{(n)}\right)\mathrm{d}W_s^{(n)} = \int_0^\cdot P_nB\left(s, X_t^{(n)}\right)\tilde{P}_n\,\mathrm{d}W_s$$

holds for all $n\in\mathbb{N}$. The mapping

$$\mathrm{Int}_W\colon \mathfrak{L}^2 \to L^2([0,T]\times\Omega; H), \quad \Phi \mapsto \mathrm{Int}_W(\Phi) := \int \Phi\,\mathrm{d}W$$

is linear and continuous, so it preserves weak convergence. Hence

$$\int_0^\cdot P_{n'''_k}B\left(s, X_t^{(n'''_k)}\right)\mathrm{d}W_s^{(n'''_k)} = \int_0^\cdot P_{n'''_k}B\left(s, X_s^{(n'''_k)}\right)\tilde{P}_{n'''_k}\,\mathrm{d}W_s \to \int_0^\cdot Z_s\,\mathrm{d}W_s$$

weakly as $k \to \infty$.

Part (iv). If we identify

$$\mathfrak{M} = L^2 \left(\Omega \times [0,T] \times Z, \mathcal{P} \otimes \mathcal{Z}, P \times \mathrm{d}t \times m; H\right),$$

then we see that $\mathfrak{M}$ is reflexive, too. The proof of boundedness of $\left(P_{n_k'''} f\left(\cdot, X^{(n_k''')}, \cdot\right)\right)_{k\in\mathbb{N}}$ is also done with (B1):

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \left(\mathbb{E}\left[\int_0^T \int_Z \left\| P_{n_k'''} f\left(t, X_{t-}^{(n_k''')}, z\right)\right\|_H^2 m(\mathrm{d}z)\,\mathrm{d}t\right]\right) \\
\overset{(B1)}{\leqslant} &\sup_{k\in\mathbb{N}} \left(\mathbb{E}\left[\int_0^T \left(C\left(1 + F_t + \left\|X_t^{(n_k''')}\right\|_H^2\right) + \gamma \left\|X_t^{(n_k''')}\right\|_V^\alpha\right) \mathrm{d}t\right]\right) \\
<&\infty \quad \text{cf. (2.3.21).}
\end{aligned}$$

So there exists a subsequence $(\bar{n}_k)_{k\in\mathbb{N}} \subset (n_k''')_{k\in\mathbb{N}}$ which fulfills (i)-(iv). Especially there is an element $g \in \mathcal{M}$ that is the weakly limit of $P_{\bar{n}_k} f\left(\cdot, X_s^{(\bar{n}_k)}, \cdot\right)$ as $k \to \infty$. The second part of (iv) follows identically as in the proof before: Since the mapping

$$\mathrm{Int}_{\bar{\mu}} \colon \mathfrak{M} \to L^2 \left([0,T] \times \Omega; H\right), \quad \Phi \mapsto \mathrm{Int}_{\bar{\mu}}(\Phi) := \int_0^\cdot \int_Z \Phi(s,z)\, \bar{\mu}(\mathrm{d}s, \mathrm{d}z)$$

is linear and continuous, it preserves weak convergence and we obtain that

$$\int_0^\cdot \int_Z P_{\bar{n}_k} f\left(s, X_{s-}^{(\bar{n}_k)}, z\right) \bar{\mu}(\mathrm{d}s, \mathrm{d}z) \to \int_0^\cdot g(s,z)\, \bar{\mu}(\mathrm{d}s, \mathrm{d}z)$$

weakly as $k \to \infty$.

□

2.3.8 Remark. *In the situation of Lemma 2.3.7 all the* $\mathrm{d}t \otimes P$*-versions* $\bar{X}$*,* Y *and* Z *are progressively measurable, since the approximants are progressively measurable.*

2.3.2. Construction of the infinite dimensional Solution

Let us recall what we have achieved so far. By the Galerkin approximation we considered the following stochastic partial differential equation in the finite dimensional space H_n, $n \in \mathbb{N}$:

$$\begin{aligned}
\mathrm{d}Y(t) =& P_n A(t, Y(t))\,\mathrm{d}t + P_n B(t, Y(t))\,\mathrm{d}W_t^{(n)} \\
&+ \int_Z P_n f(t, Y(t-), z)\, \bar{\mu}(\mathrm{d}t, \mathrm{d}z), \qquad (2.3.22)\\
Y(0) =& P_n X_0,
\end{aligned}$$

By Lemma 2.3.1 this equation has a unique strong solution $\left(X_t^{(n)}\right)_{t\in[0,T]}$ for each $n \in \mathbb{N}$. Each solution fulfills some apriori estimates from Lemma 2.3.5 which allowed us to find limiting elements $\bar{X}$, Y, Z and g as in Lemma 2.3.7 for the sequence of solutions $\left(X^{(n)}\right)_{n\in\mathbb{N}}$.

Now we come back to our origin equation

$$\begin{aligned} \mathrm{d}X(t) =& A(t, X(t))\,\mathrm{d}t + B(t, X(t))\,\mathrm{d}W(t) + \int_Z f(t, X(t-), z)\,\bar{\mu}(\mathrm{d}t, \mathrm{d}z), \\ X(0) =& X_0, \end{aligned}$$

$t \in [0, T]$. Let $\bar{X}$, Y, Z, g be as in Lemma 2.3.7. We can define the following stochastic process:

$$X(t) = X_0 + \int_0^t Y(s)\,\mathrm{d}s + \int_0^t Z(s)\,\mathrm{d}W(s) + \int_0^t \int_Z g(s, z)\,\bar{\mu}(\mathrm{d}s, \mathrm{d}z), \tag{2.3.23}$$

$t \in [0, T]$. In the following we will see that this process is a V^*-valued modification of the V-valued process $\bar{X}$ and that this process is a solution to our equation (2.1.1) which finishes the proof of uniqueness.

Notation. *For abbreviation we set*

$$Y^{(n_k)} := P_{n_k} A\left(\cdot, X^{(n_k)}\right), \quad Z^{(n_k)} := P_{n_k} B\left(\cdot, X^{(n_k)}\right), \quad f^{(n_k)} := P_{n_k} f\left(\cdot, X^{(n_k)}, \cdot\right).$$

2.3.9 Lemma. *The stochastic process $(X_t)_{t\in[0,T]}$ defined by (2.3.23) is a V^*-valued modification of $\bar{X}$.*

Proof. This proof is a straightforwarded extension of the proof of [PR07, Theorem 4.2.4, p. 86]. We have to show $X = \bar{X}$ $\mathrm{d}t \otimes P$ almost everywhere in V.
Let $v \in \bigcup_{n\geqslant 1} H_n$ $(\subset V)$ and $\varphi \in L^\infty([0, T] \times \Omega)$. Using Lemma 2.3.7 (i) and then equation (2.3.22) and Fubini's theorem we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T {}_{V^*}\langle \bar{X}(t), \varphi(t) v\rangle_V\,\mathrm{d}t\right] = \lim_{k\to\infty} \mathbb{E}\left[\int_0^T {}_{V^*}\left\langle X^{(n_k)}(t), \varphi(t) v\right\rangle_V \mathrm{d}t\right] \\
&= \lim_{k\to\infty}\Bigg(\mathbb{E}\left[\int_0^T {}_{V^*}\langle P_{n_k} X_0, \varphi(t) v\rangle_V\,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T \int_0^t {}_{V^*}\left\langle Y^{(n_k)}(s), \varphi(t) v\right\rangle_V \mathrm{d}s\,\mathrm{d}t\right] \\
&\qquad + \mathbb{E}\left[\int_0^T \left\langle \int_0^t Z^{(n_k)}(s)\,\mathrm{d}W^{(n_k)}(s), \varphi(t) v\right\rangle_H \mathrm{d}t\right] \\
&\qquad + \mathbb{E}\left[\int_0^T \left\langle \int_0^t \int_Z f^{(n_k)}(s, z)\,\bar{\mu}(\mathrm{d}s, \mathrm{d}z), \varphi(t) v\right\rangle_H \mathrm{d}t\right]\Bigg) \\
&= \lim_{k\to\infty}\Bigg(\mathbb{E}\left[\langle P_{n_k} X_0, v\rangle_H \int_0^T \varphi(t)\,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T {}_{V^*}\left\langle Y^{(n_k)}(s), \int_s^T \varphi(t)\,\mathrm{d}t\, v\right\rangle_V \mathrm{d}s\right] \\
&\qquad + \int_0^T \mathbb{E}\left[\left\langle \int_0^t Z^{(n_k)}(s)\,\mathrm{d}W^{(n_k)}(s), \varphi(t) v\right\rangle_H\right] \mathrm{d}t \\
&\qquad + \int_0^T \mathbb{E}\left[\left\langle \int_0^t \int_Z f^{(n_k)}(s, z)\,\bar{\mu}(\mathrm{d}s, \mathrm{d}z), \varphi(t) v\right\rangle_H\right] \mathrm{d}t\Bigg)
\end{aligned}$$

On the right hand side of the above equation we can now use our weak convergence results from Lemma 2.3.7 (ii)-(iv) and this yields to

$$\begin{aligned}
&\lim_{k\to\infty}\Bigg(\mathbb{E}\left[\langle P_{n_k}X_0, v\rangle_H \int_0^T \varphi(t)\,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T {}_{V^*}\left\langle Y^{(n_k)}(s), \int_s^T \varphi(t)\,\mathrm{d}t\, v\right\rangle_V \mathrm{d}s\right]\\
&\quad + \int_0^T \mathbb{E}\left[\left\langle \int_0^t Z^{(n_k)}(s)\,\mathrm{d}W^{(n_k)}(s), \varphi(t)\, v\right\rangle_H\right]\mathrm{d}t\\
&\quad + \int_0^T \mathbb{E}\left[\left\langle \int_0^t \int_Z f^{(n_k)}(s,z)\,\bar{\mu}(\mathrm{d}s,\mathrm{d}z), \varphi(t)\, v\right\rangle_H\right]\mathrm{d}t\Bigg)\\
&= \mathbb{E}\left[\int_0^T {}_{V^*}\left\langle X_0 + \int_0^t Y(s)\,\mathrm{d}s + \int_0^t Z(s)\,\mathrm{d}W(s) + \int_0^t\int_Z g(s,z)\,\bar{\mu}(\mathrm{d}s,\mathrm{d}z), \varphi(t)\, v\right\rangle_V \mathrm{d}t\right]\\
&= \mathbb{E}\left[\int_0^T {}_{V^*}\langle X(t), \varphi(t)\, v\rangle_V\,\mathrm{d}t\right],
\end{aligned}$$

what was to be shown.

□

2.3.10 Proposition. *The stochastic process* $(X_t)_{t\in[0,T]}$ *defined by (2.3.23)*

(i) is H-valued, càdlàg, $(\mathcal{F}_t)$-adapted and satisfies P-a.s. the following Itô-formula:

$$\begin{aligned}
\|X_t\|_H^2 = \|X_0\|_H^2 &+ \int_0^t \left(2\,{}_{V^*}\langle Y_s, \bar{X}_s\rangle_V\right)\mathrm{d}s\\
&+ \int_0^t \|Z_s\|_{L_2}^2\,\mathrm{d}s + \int_0^t\int_Z \|g(s,z)\|_H^2\,\mu(\mathrm{d}s,\mathrm{d}z) + 2M(t)
\end{aligned}$$

for $t \in [0,T]$, where

$$M(t) = \int_0^t \langle \bar{X}_{s-}, Z_s\,\mathrm{d}W_s\rangle_H + \int_0^t\int_Z \langle \bar{X}_{s-}, g(s,z)\rangle_H\,\bar{\mu}(\mathrm{d}s,\mathrm{d}z)$$

is a càdlàg, real-valued, local martingale.

(ii) fulfills

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^2\right] < \infty.$$

Proof. This proof is inspired by [Ste12, Theorem 5.9].
Step (i).1. Let us apply [GK82, Theorem 2] with

$$h(t) := \int_0^t Z(s)\,\mathrm{d}W(s) + \int_0^t\int_Z g(s,z)\,\bar{\mu}(\mathrm{d}s,\mathrm{d}z)$$

and $V(s) = s$. Then X defined by (2.3.23) is H-valued, càdlàg, $(\mathcal{F}_t)$-adapted and we have P-a.s.

$$\begin{aligned}\|X_t\|_H^2 = \quad & \|X_0\|_H^2 + \int_0^t 2\, {}_{V^*}\langle Y_s, \bar{X}_s\rangle_V \,\mathrm{d}s \\ & + \int_0^t \langle X_{s-}, \mathrm{d}h_s\rangle_H + [h]_t\end{aligned}$$

for all $t \in [0,T]$, where $[h]_t$ denotes the square bracket of h (see Definition D.3) and we already used that X is a V^*-valued modification of $\bar{X}$ by Lemma 2.3.9.
Step (i).2. Since X_{t-} as an $(\mathcal{F}_t)$-adapted càdlàg process is predicatble and since $Z \in \mathcal{N}_W(0,T)$, we know that the stochastic integral $\int_0^t \langle X_{s-}, Z_s \,\mathrm{d}W_s\rangle_H$ is a real-valued local martingale.
Furthermore, since $g \in \mathcal{N}_{\bar{\mu}}(T, Z; H)$, we obtain that

$$\int_0^t \int_Z \langle X_{s-}, g(s,z)\rangle_H \,\bar{\mu}(\mathrm{d}s, \mathrm{d}z) =: N_t$$

is a real-valued local martingale if we stop it by

$$\tau_n := \inf\{t \geqslant 0 \,|\, N_t > n\} \wedge T$$

for $n \in \mathbb{N}$, since we have $\lim_{n\to\infty} \tau_n = T$ P-a.s.
From this we see that the stochastic integral

$$\int_0^t \langle X_{s-}, \mathrm{d}h_s\rangle_H = \int_0^t \langle X_{s-}, Z_s \,\mathrm{d}W_s\rangle_H + \int_0^t \int_Z \langle X_{s-}, g(s,z)\rangle_H \,\bar{\mu}(\mathrm{d}s, \mathrm{d}z)$$

is a real-valued, càdlàg, local martingale.
Step (i).3. We know that

$$\left[\int_0^{\cdot} Z_s \,\mathrm{d}W_s\right]_t = \int_0^t \|Z_s\|_{L_2}^2 \,\mathrm{d}s$$

and by Proposition 1.2.8

$$\left[\int_0^{\cdot} \int_Z g(s,z)\,\bar{\mu}(\mathrm{d}s, \mathrm{d}z)\right]_t = \int_0^t \int_Z \|g(s,z)\|_H^2 \,\mu(\mathrm{d}s, \mathrm{d}z),$$

so we conclude

$$[h]_t = \int_0^t \|Z_s\|_{L_2}^2 \,\mathrm{d}s + \int_0^t \int_Z \|g(s,z)\|_H^2 \,\mu(\mathrm{d}s, \mathrm{d}z).$$

Now we want to prove (ii).
Step (ii).1. We define

$$\tau_R := \inf\left\{t \geqslant 0 \,\middle|\, \left\|\bar{X}_t\right\|_H > R\right\} \wedge T,$$

which is a stopping time due to Theorem D.4 and use the notation

$$\hat{t} := \hat{t}\,(t, R) = t \wedge \tau_R$$

for $t \in [0, T]$ and $R > 0$. We have $\tau_R \to T$ for $R \to \infty$ P-a.s. Itô's formula from 2.3.10 (i) applied with Hölder's inequality yields to

$$\begin{aligned}\|X_t\|_H^2 \leqslant \|X_0\|_H^2 &+ \left(\int_0^T \|Y_s\|_{V^*}^{\frac{\alpha}{\alpha-1}}\,\mathrm{d}s\right)^{\frac{\alpha-1}{\alpha}} \left(\int_0^T \|\bar{X}_s\|_V^\alpha\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \\ &+ \int_0^T \|Z_s\|_{L_2}^2\,\mathrm{d}s + \int_0^T \int_Z \|g\,(s,z)\|_H^2\,\mu\,(\mathrm{d}s, \mathrm{d}z) \\ &+ 2\left|\int_0^t \left\langle \bar{X}_{s-}, Z_s\,\mathrm{d}W_s\right\rangle_H\right| + 2\left|\int_0^t \int_Z \left\langle \bar{X}_{s-}, g\,(s,z)\right\rangle_H \bar{\mu}\,(\mathrm{d}s, \mathrm{d}z)\right|.\end{aligned}$$

Now we take the supremum over $\left[0, \hat{t}\right]$ and then apply the expectation to both sides:

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\in[0,\hat{t}]} \|X_s\|_H^2\right] \\ &\leqslant \mathbb{E}\left[\|X_0\|_H^2\right] + \mathbb{E}\left[\left(\int_0^T \|Y_s\|_{V^*}^{\frac{\alpha}{\alpha-1}}\,\mathrm{d}s\right)^{\frac{\alpha-1}{\alpha}} \left(\int_0^T \|\bar{X}_s\|_V^\alpha\,\mathrm{d}s\right)^{\frac{1}{\alpha}}\right] \\ &+ \mathbb{E}\left[\int_0^T \|Z_s\|_{L_2}^2\,\mathrm{d}s\right] + \mathbb{E}\left[\int_0^T \int_Z \|g\,(s,z)\|_H^2\,\mu\,(\mathrm{d}s, \mathrm{d}z)\right] \\ &+ 2\,\mathbb{E}\left[\sup_{r\in[0,\hat{t}]}\left|\int_0^r \left\langle \bar{X}_{s-}, Z_s\,\mathrm{d}W_s\right\rangle_H\right|\right] + 2\,\mathbb{E}\left[\sup_{r\in[0,\hat{t}]}\left|\int_0^r \int_Z \left\langle \bar{X}_{s-}, g\,(s,z)\right\rangle_H \bar{\mu}\,(\mathrm{d}s, \mathrm{d}z)\right|\right].\end{aligned} \tag{2.3.24}$$

Step (ii).2. We apply the Burkholder-Davis-Gundy inequality D.5 (i) on the first summand in the last row of (2.3.24) and Lemma 2.3.9

$$\begin{aligned}&2\,\mathbb{E}\left[\sup_{r\in[0,\hat{t}]}\left|\int_0^r \left\langle \bar{X}_{s-}, Z_s\,\mathrm{d}W_s\right\rangle_H\right|\right] \\ &\overset{D.5\,(i)}{\leqslant} 2C_{BDG}\,\mathbb{E}\left[\left[\int_0^r \left\langle \bar{X}_{s-}, Z_s\,\mathrm{d}W_s\right\rangle_H\right]_{\hat{t}}^{\frac{1}{2}}\right] \leqslant 2C_{BDG}\,\mathbb{E}\left[\left(\int_0^{\hat{t}} \|\bar{X}_{s-}\|_H^2\,\|Z_s\|_{L_2}^2\,\mathrm{d}s\right)^{\frac{1}{2}}\right] \\ &\overset{2.3.9}{=} 2C_{BDG}\,\mathbb{E}\left[\left(\int_0^{\hat{t}} \|X_{s-}\|_H^2\,\|Z_s\|_{L_2}^2\,\mathrm{d}s\right)^{\frac{1}{2}}\right].\end{aligned}$$

For $\varepsilon > 0$ and by Young's inequality we get

$$
\begin{aligned}
2\,\mathbb{E}\left[\sup_{r\in[0,\hat{t}]}\left|\int_0^r \left\langle \bar{X}_{s-}, Z_s\,\mathrm{d}W_s\right\rangle_H\right|\right] \leqslant & 2C_{BDG}\,\mathbb{E}\left[\left(\int_0^{\hat{t}} \|X_{s-}\|_H^2\,\|Z_s\|_{L_2}^2\,\mathrm{d}s\right)^{\frac{1}{2}}\right]\\
\leqslant & 2C_{BDG}\,\mathbb{E}\left[\left(\varepsilon\sup_{s\in[0,\hat{t}]}\|X_{s-}\|_H^2\right)^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\int_0^{\hat{t}}\|Z_s\|_{L_2}^2\,\mathrm{d}s\right)^{\frac{1}{2}}\right]\\
\overset{Young}{\leqslant} & 2C_{BDG}\,\mathbb{E}\left[\frac{\varepsilon}{2}\sup_{s\in[0,\hat{t}]}\|X_s\|_H^2+\frac{1}{2\varepsilon}\int_0^T\|Z_s\|_{L_2}^2\,\mathrm{d}s\right]\\
= & C_{BDG}\varepsilon\,\mathbb{E}\left[\sup_{s\in[0,\hat{t}]}\|X_s\|_H^2\right]+\frac{C_{BDG}}{\varepsilon}\,\mathbb{E}\left[\int_0^T\|Z_s\|_{L_2}^2\,\mathrm{d}s\right]
\end{aligned}
$$

and we used $\sup_{s\in[0,t]}\|X_{s-}\|_H^2 \leqslant \sup_{s\in[0,t]}\|X_s\|_H^2$ here.
Step (ii).3. Again, the Burkholder-Davis-Gundy inequality D.5 (i) applied to the second term in the last row of (2.3.24) and then Proposition 1.2.8 together with the Cauchy-Schwarz inequality yields to

$$
\begin{aligned}
& 2\,\mathbb{E}\left[\sup_{r\in[0,\hat{t}]}\left|\int_0^r\int_Z \left\langle \bar{X}_{s-}, g(s,z)\right\rangle_H\,\bar{\mu}(\mathrm{d}s,\mathrm{d}z)\right|\right]\\
\overset{D.5\,(i)}{\leqslant} & 2C_{BDG}\,\mathbb{E}\left[\left[\int_0^{\cdot}\int_Z\left\langle \bar{X}_{s-}, g(s,z)\right\rangle_H\,\bar{\mu}(\mathrm{d}s,\mathrm{d}z)\right]_{\hat{t}}^{\frac{1}{2}}\right]\\
\overset{1.2.8}{\leqslant} & 2C_{BDG}\,\mathbb{E}\left[\left(\int_0^{\hat{t}}\int_Z\|\bar{X}_{s-}\|_H^2\,\|g(s,z)\|_H^2\,\mu(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{1}{2}}\right]\\
\leqslant & 2C_{BDG}\,\mathbb{E}\left[\left(\sup_{s\in[0,\hat{t}]}\|X_s\|_H^2\int_0^{\hat{t}}\int_Z\|g(s,z)\|_H^2\,\mu(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{1}{2}}\right],
\end{aligned}
$$

where we used that $X = \bar{X}$ $\mathrm{d}t\otimes P$-a.e. by Lemma 2.3.9 and that $\sup_{s\in[0,t]}\|X_{s-}\|_H^2 \leqslant \sup_{s\in[0,t]}\|X_s\|_H^2$ in the last step. Let $\varepsilon > 0$. Proposition 1.2.7 and Young's inequality leads to

$$
\begin{aligned}
&2C_{BDG}\,\mathbb{E}\left[\left(\sup_{s\in[0,\hat{t}]}\|X_s\|_H^2\int_0^{\hat{t}}\int_Z\|g(s,z)\|_H^2\,\mu(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{1}{2}}\right]\\
&\overset{1.2.7}{=}2C_{BDG}\,\mathbb{E}\left[\left(\varepsilon\sup_{s\in[0,\hat{t}]}\|X_s\|_H^2\right)^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\int_0^{\hat{t}}\int_Z\|g(s,z)\|_H^2\,m(\mathrm{d}s)\,\mathrm{d}s\right)^{\frac{1}{2}}\right]\\
&\leqslant C_{BDG}\varepsilon\,\mathbb{E}\left[\sup_{s\in[0,\hat{t}]}\|X_s\|_H^2\right]+\frac{C_{BDG}}{\varepsilon}\,\mathbb{E}\left[\int_0^{\hat{t}}\int_Z\|g(s,z)\|_H^2\,m(\mathrm{d}s)\,\mathrm{d}s\right].
\end{aligned}
$$

Step (ii).4. Combining the results from the last two steps for $\varepsilon=\frac{1}{4C_{BDG}}$ and inserting them into (2.3.24) yields to

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}\left[\sup_{s\in[0,\hat{t}]}\|X_s\|_H^2\right]\leqslant{}&\mathbb{E}\left[\|X_0\|_H^2\right]+\mathbb{E}\left[\left(\int_0^T\|Y_s\|_{V^*}^{\frac{\alpha}{\alpha-1}}\,\mathrm{d}s\right)^{\frac{\alpha-1}{\alpha}}\left(\int_0^T\|\bar{X}_s\|_V^{\alpha}\,\mathrm{d}s\right)^{\frac{1}{\alpha}}\right]\\
&+4C_{BDG}^2\left(\mathbb{E}\left[\int_0^T\|Z_s\|_{L_2}^2\,\mathrm{d}s\right]+\mathbb{E}\left[\int_0^T\int_Z\|g(s,z)\|_H^2\,\mu(\mathrm{d}s,\mathrm{d}z)\right]\right)
\end{aligned}
\tag{2.3.25}
$$

Since the right hand side is independent of R, we only have to show that it is finite. For $p>2$ (if $p=2$ there is nothing to show) we infer with Hölder's inequality

$$
\mathbb{E}\left[\|X_0\|_H^2\right]\leqslant\mathbb{E}\left[\|X_0\|_H^p\right]^{\frac{2}{p}}\cdot\mathbb{E}\left[1\right]^{\frac{p-2}{p}}=\mathbb{E}\left[\|X_0\|_H^p\right]^{\frac{2}{p}}<\infty.
$$

Since $\bar{X}\in\mathfrak{L}^{\alpha}$ by Lemma 2.3.7 (i) and since $Y\in\mathfrak{L}^{\alpha'}$ by Lemma 2.3.7 (ii) we see with Young's inequality for $q=\frac{\alpha}{\alpha-1}$, $q'=\alpha$

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T\|Y_s\|_{V^*}^{\frac{\alpha}{\alpha-1}}\,\mathrm{d}s\right)^{\frac{\alpha-1}{\alpha}}\left(\int_0^T\|\bar{X}_s\|_V^{\alpha}\,\mathrm{d}s\right)^{\frac{1}{\alpha}}\right]\\
&\leqslant\frac{\alpha-1}{\alpha}\,\mathbb{E}\left[\int_0^T\|Y_s\|_{V^*}^{\frac{\alpha}{\alpha-1}}\,\mathrm{d}s\right]+\frac{1}{\alpha}\,\mathbb{E}\left[\int_0^T\|\bar{X}_s\|_V^{\alpha}\,\mathrm{d}s\right]<\infty.
\end{aligned}
$$

From Lemma 2.3.7 (iii) we have $Z\in\mathfrak{L}^2$ and therefore

$$
\mathbb{E}\left[\int_0^T\|Z_s\|_{L_2}^2\,\mathrm{d}s\right]<\infty
$$

and, by Proposition 1.2.7 and Lemma 2.3.7 (iv), we see that

$$
\mathbb{E}\left[\int_0^T\int_Z\|g(s,z)\|_H^2\,\mu(\mathrm{d}s,\mathrm{d}z)\right]\overset{1.2.7}{=}\mathbb{E}\left[\int_0^T\int_Z\|g(s,z)\|_H^2\,m(\mathrm{d}z)\,\mathrm{d}s\right]<\infty.
$$

Hence

$$\mathbb{E}\left[\sup_{s\in[0,\hat{t}]}\|X_s\|_H^2\right]<\infty$$

independent of R. Letting $R\to\infty$, we finish the proof.

□

2.3.11 Corollary. *In the situation of Proposition 2.3.10, the appearing local martingale*

$$\int_0^t \left\langle \bar{X}_{s-}, Z_s\,\mathrm{d}W_s\right\rangle_H + \int_0^t\int_Z \left\langle \bar{X}_{s-}, g(s,z)\right\rangle_H \bar{\mu}(\mathrm{d}s,\mathrm{d}z)$$

is a (global) martingale.

Proof. We use the Cauchy-Schwarz inequality, Lemma 2.3.9 and Hölder's inequality on

$$\mathbb{E}\left[\left[\int_0^\cdot \left\langle \bar{X}_{s-}, Z_s\,\mathrm{d}W_s\right\rangle_H\right]_T^{\frac{1}{2}}\right] \leqslant \mathbb{E}\left[\left(\int_0^T \|X_{s-}\|_H^2\,\|Z_s\|_{L_2}^2\,\mathrm{d}s\right)^{\frac{1}{2}}\right]$$

$$\leqslant \mathbb{E}\left[\left(\sup_{t\in[0,T]}\|X_s\|_H^2\right)^{\frac{1}{2}}\left(\int_0^T\|Z_s\|_{L_2}^2\,\mathrm{d}s\right)^{\frac{1}{2}}\right]$$

$$\overset{\text{Hölder}}{\leqslant} \mathbb{E}\left[\sup_{t\in[0,T]}\|X_s\|_H^2\right]^{\frac{1}{2}}\mathbb{E}\left[\int_0^T\|Z_s\|_{L_2}^2\,\mathrm{d}s\right]^{\frac{1}{2}}$$

$$<\infty$$

by Proposition 2.3.10 (ii) and $Z\in\mathfrak{L}^2$. Doing the same together with Proposition 1.2.7 yields to

$$\mathbb{E}\left[\left[\int_0^\cdot\int_Z \left\langle \bar{X}_{s-}, g(s,z)\right\rangle_H \bar{\mu}(\mathrm{d}s,\mathrm{d}z)\right]_T^{\frac{1}{2}}\right]$$

$$\leqslant \mathbb{E}\left[\left(\int_0^T\int_Z \|\bar{X}_{s-}\|_H^2\,\|g(s,z)\|_H^2\,\mu(\mathrm{d}s,\mathrm{d}z)\right)^{\frac{1}{2}}\right]$$

$$\leqslant \mathbb{E}\left[\left(\sup_{t\in[0,T]}\|X_s\|_H^2\right)^{\frac{1}{2}}\left(\int_0^T\int_Z\|g(s,z)\|_H^2\,m(\mathrm{d}z)\,\mathrm{d}s\right)^{\frac{1}{2}}\right]$$

$$\leqslant \mathbb{E}\left[\sup_{t\in[0,T]}\|X_s\|_H^2\right]^{\frac{1}{2}}\mathbb{E}\left[\int_0^T\int_Z\|g(s,z)\|_H^2\,m(\mathrm{d}z)\,\mathrm{d}s\right]^{\frac{1}{2}}<\infty,$$

since $g\in\mathfrak{M}$ and Proposition 2.3.10 (ii).

Now, using Theorem D.5 (ii), we see that

$$\int_0^t \left\langle \bar{X}_{s-}, Z_s \,\mathrm{d}W_s \right\rangle_H + \int_0^t \int_Z \left\langle \bar{X}_{s-}, g\left(s,z\right) \right\rangle_H \bar{\mu}\left(\mathrm{d}s, \mathrm{d}z\right)$$

is a global $L^1\left(\Omega;\mathbb{R}\right)$-martingale.

□

The following Proposition finishes the proof of existence.

2.3.12 Proposition. *In the above situation we have*

$$\begin{aligned} A\left(\cdot,\bar{X}\right) &= Y && \mathrm{d}t\otimes P\text{-a.e.},\\ B\left(\cdot,\bar{X}\right) &= Z && \mathrm{d}t\otimes P\text{-a.e.},\\ f\left(s,\bar{X}_{s-},z\right) &= g\left(s,z\right) && \mathrm{d}t\otimes P\otimes m\text{-a.e.} \end{aligned}$$

Therefore the process $(X_t)_{t\in[0,T]}$ *given by (2.3.23) is a solution to (2.1.1).*

Proof. Define

$$\mathfrak{N} := \mathfrak{L}^\alpha \cap L^p\left(\Omega; L^\infty\left([0,T];H\right)\right).$$

Let ϕ be a V-valued, progressively measurable version of an element in $\mathfrak{N}$ such that

$$\mathbb{E}\left[\int_0^T \varrho\left(\phi\right)\,\mathrm{d}s\right] < \infty.$$

In the following we will see that the integral $\int_0^t \left(F\left(s\right) + \varrho\left(\phi\left(s\right)\right)\right)\mathrm{d}s$ has to be finite for our calculations. Since $V \hookrightarrow H$ continuously and by our estimate of ϱ in condition (B3), the following definition is justified: Let $\tau^\phi \colon \Omega \to [0,T]$ defined by

$$\tau^\phi := \tau^\phi\left(\phi, R\right) := \inf\left\{0 \leqslant t \leqslant T \,\middle|\, \int_0^t \left(F\left(s\right) + \left\|\phi\left(s\right)\right\|_V^\alpha\right)\mathrm{d}s > R\right\} \wedge T$$

for $R > 0$. Then τ^ϕ is a stopping time. For simplicity we set

$$\hat{t} := \hat{t}\left(t,\phi,R\right) := t \wedge \tau^\phi, \quad 0 \leqslant t \leqslant T.$$

Step 1. Applying Itô's formula together with Itô's product rule we obtain for $t \in [0,T]$

$$\begin{aligned} &\mathbb{E}\left[e^{-\int_0^{\hat{t}}\left(F_s+\varrho(\phi_s)\right)\mathrm{d}s}\left\|X_{\hat{t}}^{(n_k)}\right\|_H^2\right] - \mathbb{E}\left[\left\|X_0^{(n_k)}\right\|_H^2\right]\\ =\; &\mathbb{E}\left[\int_0^{\hat{t}} e^{-\int_0^s\left(F_r+\varrho(\phi_r)\right)\mathrm{d}r}\left(2\,{}_{V^*}\!\left\langle A\left(s,X_s^{(n_k)}\right), X_{s-}^{(n_k)}\right\rangle_V + \left\|P_{n_k}B\left(s,X_s^{(n_k)}\right)\tilde{P}_{n_k}\right\|_{L_2}^2\right.\right.\\ &\qquad\left.\left. - \left(F_s + \varrho\left(\phi_s\right)\right)\left\|X_s^{(n_k)}\right\|_H^2\right)\mathrm{d}s\right]\\ &+\mathbb{E}\left[\int_0^{\hat{t}}\int_Z e^{-\int_0^s\left(F_r+\varrho(\phi_r)\right)\mathrm{d}r}\left\|P_{n_k}f\left(s,X_{s-}^{(n_k)},z\right)\right\|_H^2 \mu\left(\mathrm{d}z,\mathrm{d}s\right)\right], \end{aligned}$$

where we used that the expectation of the appearing martingale is zero. Proposition 1.2.7 gives us

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\int_0^{\hat{t}}(F_s+\varrho(\phi_s))\mathrm{d}s}\left\|X_{\hat{t}}^{(n_k)}\right\|_H^2\right]-\mathbb{E}\left[\left\|X_0^{(n_k)}\right\|_H^2\right]\\
=&\mathbb{E}\left[\int_0^{\hat{t}}e^{-\int_0^s(F_r+\varrho(\phi_r))\mathrm{d}r}\left(2\,{}_{V^*}\left\langle A\left(s,X_s^{(n_k)}\right),X_{s-}^{(n_k)}\right\rangle_V+\left\|P_{n_k}B\left(s,X_s^{(n_k)}\right)\tilde{P}_{n_k}\right\|_{L_2}^2\right.\right.\\
&\left.\left.-\left(F_s+\varrho\left(\phi_s\right)\right)\left\|X_s^{(n_k)}\right\|_H^2+\int_Z\left\|P_{n_k}f\left(s,X_{s-}^{(n_k)},z\right)\right\|_H^2m\left(\mathrm{d}z\right)\right)\mathrm{d}s\right]\\
\leqslant&\mathbb{E}\left[\int_0^{\hat{t}}e^{-\int_0^s(F_r+\varrho(\phi_r))\mathrm{d}r}\left(2\,{}_{V^*}\left\langle A\left(s,X_s^{(n_k)}\right),X_s^{(n_k)}\right\rangle_V+\left\|B\left(s,X_s^{(n_k)}\right)\right\|_{L_2}^2\right.\right.\\
&\left.\left.-\left(F_s+\varrho\left(\phi_s\right)\right)\left\|X_s^{(n_k)}\right\|_H^2+\int_Z\left\|f\left(s,X_s^{(n_k)},z\right)\right\|_H^2m\left(\mathrm{d}z\right)\right)\mathrm{d}s\right].
\end{aligned}
\tag{2.3.26}
$$

In the last step we made the norms bigger by leaving out the projection P_{n_k}.
Step 2. It is helpful in the following calculations to have in mind that by the defintion of the inner product we can calculate

$$
\begin{aligned}
{}_{V^*}\left\langle A\left(s,X_s^{(n_k)}\right),X_s^{(n_k)}\right\rangle_V=&\ {}_{V^*}\left\langle A\left(s,X_s^{(n_k)}\right)-A\left(s,\phi_s\right),X_s^{(n_k)}-\phi_s\right\rangle_V\\
&+{}_{V^*}\left\langle A\left(s,\phi_s\right),X_s^{(n_k)}\right\rangle_V+{}_{V^*}\left\langle A\left(s,X_s^{(n_k)}\right)-A\left(s,\phi_s\right),\phi_s\right\rangle_V,
\end{aligned}
$$

$$
\begin{aligned}
\left\|B\left(s,X_s^{(n_k)}\right)\right\|_{L_2}^2=&\left\|B\left(s,X_s^{(n_k)}\right)-B\left(s,\phi_s\right)\right\|_{L_2}^2-\left\|B\left(s,\phi_s\right)\right\|_{L_2}^2\\
&+2\left\langle B\left(s,X_s^{(n_k)}\right),B\left(s,\phi_s\right)\right\rangle_{L_2},
\end{aligned}
$$

$$
\begin{aligned}
\left\|f\left(s,X_s^{(n_k)},z\right)\right\|_H^2=&\left\|f\left(s,X_s^{(n_k)},z\right)-f\left(s,\phi_s,z\right)\right\|_H^2-\left\|f\left(s,\phi_s,z\right)\right\|_H^2\\
&+2\left\langle f\left(s,X_s^{(n_k)},z\right),f\left(s,\phi_s,z\right)\right\rangle_H
\end{aligned}
$$

and

$$
\left\|X_s^{(n_k)}-\phi_s\right\|_H^2-\left\|X_s^{(n_k)}\right\|_H^2=\left\|\phi_s\right\|_H^2-2\left\langle X_s^{(n_k)},\phi_s\right\rangle_H. \tag{2.3.27}
$$

When referring to (2.3.27) we think of one of the four equations above depending on the context. Now, the local monotonicity condition (A2) gives us for

$$
\begin{aligned}
& 2\,{}_{V^*}\left\langle A\left(s, X_s^{(n_k)}\right), X_s^{(n_k)}\right\rangle_V + \left\|B\left(s, X_s^{(n_k)}\right)\right\|_{L_2}^2 + \int_Z \left\|f\left(s, X_s^{(n_k)}, z\right)\right\|_H^2 m(\mathrm{d}z) \\
= \; & 2\,{}_{V^*}\left\langle A\left(s, X_s^{(n_k)}\right) - A(s,\phi_s), X_s^{(n_k)} - \phi_s\right\rangle_V + \left\|B\left(s, X_s^{(n_k)}\right) - B(s,\phi_s)\right\|_{L_2}^2 \\
& + \int_Z \left\|f\left(s, X_s^{(n_k)}, z\right) - f(s,\phi_s,z)\right\|_H^2 m(\mathrm{d}z) \\
& + 2\,{}_{V^*}\left\langle A(s,\phi_s), X_s^{(n_k)}\right\rangle_V + 2\,{}_{V^*}\left\langle A\left(s, X_s^{(n_k)}\right) - A(s,\phi_s), \phi_s\right\rangle_V \\
& + 2\left\langle B\left(s, X_s^{(n_k)}\right), B(s,\phi_s)\right\rangle_{L_2} - \|B(s,\phi_s)\|_{L_2}^2 \\
& + \int_Z \left(2\left\langle f\left(s, X_s^{(n_k)}, z\right), f(s,\phi_s,z)\right\rangle_H - \|f(s,\phi_s,z)\|_H^2\right) m(\mathrm{d}z) \\
\overset{\text{(A2)}}{\leqslant} \; & (F_s + \varrho(\phi_s)) \left\|X_s^{(n_k)} - \phi_s\right\|_H^2 \\
& + 2\,{}_{V^*}\left\langle A(s,\phi_s), X_s^{(n_k)}\right\rangle_V + 2\,{}_{V^*}\left\langle A\left(s, X_s^{(n_k)}\right) - A(s,\phi_s), \phi_s\right\rangle_V \\
& + 2\left\langle B\left(s, X_s^{(n_k)}\right), B(s,\phi_s)\right\rangle_{L_2} - \|B(s,\phi_s)\|_{L_2}^2 \\
& + \int_Z \left(2\left\langle f\left(s, X_s^{(n_k)}, z\right), f(s,\phi_s,z)\right\rangle_H - \|f(s,\phi_s,z)\|_H^2\right) m(\mathrm{d}z).
\end{aligned}
\tag{2.3.28}
$$

Inserting (2.3.28) into (2.3.26) yields to

$$
\begin{aligned}
& \mathbb{E}\left[e^{-\int_0^{\hat{t}}(F_s + \varrho(\phi_s))\mathrm{d}s} \left\|X_{\hat{t}}^{(n_k)}\right\|_H^2\right] - \mathbb{E}\left[\left\|X_0^{(n_k)}\right\|_H^2\right] \\
\leqslant \; & \mathbb{E}\Bigg[\int_0^{\hat{t}} e^{-\int_0^s (F_r + \varrho(\phi_r))\mathrm{d}r} \Bigg((F_s + \varrho(\phi_s)) \left\|X_s^{(n_k)} - \phi_s\right\|_H^2 - (F_s + \varrho(\phi_s)) \left\|X_s^{(n_k)}\right\|_H^2 \\
& \quad + 2\,{}_{V^*}\left\langle A(s,\phi_s), X_s^{(n_k)}\right\rangle_V + 2\,{}_{V^*}\left\langle A\left(s, X_s^{(n_k)}\right) - A(s,\phi_s), \phi_s\right\rangle_V \\
& \quad + 2\left\langle B\left(s, X_s^{(n_k)}\right), B(s,\phi_s)\right\rangle_{L_2} - \|B(s,\phi_s)\|_{L_2}^2 \\
& \quad + \int_Z \left(2\left\langle f\left(s, X_s^{(n_k)}, z\right), f(s,\phi_s,z)\right\rangle_H - \|f(s,\phi_s,z)\|_H^2\right) m(\mathrm{d}z) \Bigg) \mathrm{d}s\Bigg].
\end{aligned}
\tag{2.3.29}
$$

Now, by (2.3.27), we obtain from (2.3.29)

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\hat{t}} e^{-\int_0^s (F_r+\varrho(\phi_r))\mathrm{d}r}\Bigg((F_s+\varrho(\phi_s))\left\|X_s^{(n_k)}-\phi_s\right\|_H^2 - (F_s+\varrho(\phi_s))\left\|X_s^{(n_k)}\right\|_H^2\right.\\
&\quad + 2\,{}_{V^*}\left\langle A(s,\phi_s), X_s^{(n_k)}\right\rangle_V + 2\,{}_{V^*}\left\langle A\left(s,X_s^{(n_k)}\right)-A(s,\phi_s),\phi_s\right\rangle_V\\
&\quad + 2\left\langle B\left(s,X_s^{(n_k)}\right), B(s,\phi_s)\right\rangle_{L_2} - \|B(s,\phi_s)\|_{L_2}^2\\
&\quad \left.+\int_Z \left(2\left\langle f\left(s,X_s^{(n_k)},z\right), f(s,\phi_s,z)\right\rangle_H - \|f(s,\phi_s,z)\|_H^2\right) m(\mathrm{d}z)\Bigg)\,\mathrm{d}s\right]\\
=&\mathbb{E}\left[\int_0^{\hat{t}} e^{-\int_0^s (F_r+\varrho(\phi_r))\mathrm{d}r}\Bigg((F_s+\varrho(\phi_s))\left(\|\phi_s\|_H^2 - 2\left\langle X_s^{(n_k)},\phi_s\right\rangle_H\right)\right.\\
&\quad + 2\,{}_{V^*}\left\langle A(s,\phi_s), X_s^{(n_k)}\right\rangle_V + 2\,{}_{V^*}\left\langle A\left(s,X_s^{(n_k)}\right)-A(s,\phi_s),\phi_s\right\rangle_V\\
&\quad + 2\left\langle B\left(s,X_s^{(n_k)}\right), B(s,\phi_s)\right\rangle_{L_2} - \|B(s,\phi_s)\|_{L_2}^2\\
&\quad \left.+\int_Z \left(2\left\langle f\left(s,X_s^{(n_k)},z\right), f(s,\phi_s,z)\right\rangle_H - \|f(s,\phi_s,z)\|_H^2\right) m(\mathrm{d}z)\Bigg)\,\mathrm{d}s\right].
\end{aligned}$$

Step 3. Let $\psi \in L^\infty([0,T];\mathrm{d}t)$ be non-negative. Then we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \psi_t\left(e^{-\int_0^{\hat{t}}(F_s+\varrho(\phi_s))\mathrm{d}s}\|X_t\|_H^2 - \|X_0\|_H^2\right)\mathrm{d}t\right]\\
\leqslant& \liminf_{k\to\infty}\mathbb{E}\left[\int_0^T \psi_t\left(e^{-\int_0^{\hat{t}}(F_s+\varrho(\phi_s))\mathrm{d}s}\left\|X_t^{(n_k)}\right\|_H^2 - \left\|X_0^{(n_k)}\right\|_H^2\right)\mathrm{d}t\right]\\
\leqslant& \liminf_{k\to\infty}\mathbb{E}\left[\int_0^T \psi_t\Bigg(\int_0^{\hat{t}} e^{-\int_0^s (F_r+\varrho(\phi_r))\mathrm{d}r}\Bigg((F_s+\varrho(\phi_s))\left(\|\phi_s\|_H^2 - 2\left\langle X_s^{(n_k)},\phi_s\right\rangle_H\right)\right.\\
&\quad + 2\,{}_{V^*}\left\langle A(s,\phi_s), X_s^{(n_k)}\right\rangle_V + 2\,{}_{V^*}\left\langle A\left(s,X_s^{(n_k)}\right)-A(s,\phi_s),\phi_s\right\rangle_V\\
&\quad + 2\left\langle B\left(s,X_s^{(n_k)}\right), B(s,\phi_s)\right\rangle_{L_2} - \|B(s,\phi_s)\|_{L_2}^2\\
&\quad \left.+\int_Z \left(2\left\langle f\left(s,X_s^{(n_k)},z\right), f(s,\phi_s,z)\right\rangle_H - \|f(s,\phi_s,z)\|_H^2\right) m(\mathrm{d}z)\Bigg)\,\mathrm{d}s\Bigg)\,\mathrm{d}t\right].
\end{aligned} \tag{2.3.30}$$

By Itô's formula in Proposition 2.3.10 and Itô's product rule we have for $\phi \in \mathfrak{N} \cap \mathfrak{M}$

together with Corollary 2.3.11

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_0^{\hat{t}}(F_s+\varrho(\phi_s))\mathrm{d}s}\,\|X_t\|_H^2\right]-\mathbb{E}\left[\|X_0\|_H^2\right]\\
=&\,\mathbb{E}\Bigg[\int_0^{\hat{t}} e^{-\int_0^s(F_r+\varrho(\phi_r))\mathrm{d}r}\Bigg(2\,{}_{V^*}\langle Y_s,\bar{X}_s\rangle_V+\|Z_s\|_{L_2}^2\\
&-(F_s+\varrho(\phi_s))\,\|X_s\|_H^2+\int_Z\|g(s,z)\|_H^2\;m(\mathrm{d}z)\Bigg)\,\mathrm{d}s\Bigg].
\end{aligned}\tag{2.3.31}$$

We can now insert (2.3.31) into (2.3.30) in the following way by using Fubini's theorem:

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_0^T\psi_t\Bigg(\int_0^{\hat{t}} e^{-\int_0^s(F_r+\varrho(\phi_r))\mathrm{d}r}\Bigg(2\,{}_{V^*}\langle Y_s,\bar{X}_s\rangle_V+\|Z_s\|_{L_2}^2\\
&\quad-(F_s+\varrho(\phi_s))\,\|X_s\|_H^2+\int_Z\|g(s,z)\|_H^2\;m(\mathrm{d}z)\Bigg)\,\mathrm{d}s\Bigg)\,\mathrm{d}t\Bigg]\\
\overset{(2.3.31)}{=}&\,\mathbb{E}\left[\int_0^T\psi_t\left(e^{-\int_0^{\hat{t}}(F_s+\varrho(\phi_s))\mathrm{d}s}\,\|X_t\|_H^2-\|X_0\|_H^2\right)\mathrm{d}t\right]\\
\overset{(2.3.30)}{\leqslant}&\liminf_{k\to\infty}\mathbb{E}\Bigg[\int_0^T\psi_t\Bigg(\int_0^{\hat{t}} e^{-\int_0^s(F_r+\varrho(\phi_r))\mathrm{d}r}\Bigg((K+\varrho(\phi_s))\left(\|\phi_s\|_H^2-2\left\langle X_s^{(n_k)},\phi_s\right\rangle_H\right)\\
&\quad+2\,{}_{V^*}\left\langle A(s,\phi_s),X_s^{(n_k)}\right\rangle_V+2\,{}_{V^*}\left\langle A\left(s,X_s^{(n_k)}\right)-A(s,\phi_s),\phi_s\right\rangle_V\\
&\quad+2\left\langle B\left(s,X_s^{(n_k)}\right),B(s,\phi_s)\right\rangle_{L_2}-\|B(s,\phi_s)\|_{L_2}^2\\
&\quad+\int_Z\left(2\left\langle f\left(s,X_s^{(n_k)},z\right),f(s,\phi_s,z)\right\rangle_H-\|f(s,\phi_s,z)\|_H^2\right)m(\mathrm{d}z)\Bigg)\,\mathrm{d}s\Bigg)\,\mathrm{d}t\Bigg].
\end{aligned}$$

Hence we get with the definition of the norm and inner product (cf. (2.3.27) in step 2) by bringing the terms on the right hand side to the left hand side

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_0^T\psi_t\Bigg(\int_0^{\hat{t}} e^{-\int_0^s(F_r+\varrho(\phi_r))\mathrm{d}r}\Bigg(2\,{}_{V^*}\langle Y_s-A(s,\phi_s),\bar{X}_s-\phi_s\rangle_V+\|Z_s-B(s,\phi_s)\|_{L_2}^2\\
&-(F_s+\varrho(\phi_s))\left\|\bar{X}_s-\phi_s\right\|_H^2+\int_Z\|g(s,z)-f(s,\phi_s,z)\|_H^2\;m(\mathrm{d}z)\Bigg)\,\mathrm{d}s\Bigg)\,\mathrm{d}t\Bigg]\leqslant 0.
\end{aligned}\tag{2.3.32}$$

Now we take $\phi = \bar{X}$ and get by (2.3.32) that

$$
\begin{aligned}
0 \leqslant \mathbb{E}\Bigg[\int_0^T \psi_t\Bigg(\int_0^{\hat{t}} e^{-\int_0^s (F_r+\varrho(\bar{X}_r))\mathrm{d}r}\Bigg(&\\
\left\|Z_s - B\left(s,\bar{X}_s\right)\right\|_{L_2}^2 + \int_Z \left\|g\left(s,z\right) - f\left(s,\bar{X}_s,z\right)\right\|_H^2 m\left(\mathrm{d}z\right)\Bigg)\,\mathrm{d}s\Bigg)\,\mathrm{d}t\Bigg]&\\
\leqslant 0&
\end{aligned}
$$

and letting $R \to \infty$ we see that we have by the arbitrariness of ψ

$$
\begin{aligned}
Z &= B\left(\cdot,\bar{X}\right) \quad \text{in } \mathfrak{L}^2,\\
g &= f\left(\cdot,\bar{X},\cdot\right) \quad \text{in } \mathfrak{M}.
\end{aligned}
$$

Step 4. We consider the following inequality

$$
\begin{aligned}
&\mathbb{E}\Bigg[\int_0^T \psi_t\Bigg(\int_0^{\hat{t}} e^{-\int_0^s (F_r+\varrho(\phi_r))\mathrm{d}r}\Big(2\,{}_{V^*}\langle Y_s - A\left(s,\phi_s\right), \bar{X}_s - \phi_s\rangle_V\\
&\quad - \left(F_s + \varrho\left(\phi_s\right)\right)\left\|\bar{X}_s - \phi_s\right\|_H^2\Big)\,\mathrm{d}s\Bigg)\,\mathrm{d}t\Bigg] \overset{(2.3.32)}{\leqslant} 0.
\end{aligned}
\tag{2.3.33}
$$

Let $\varepsilon > 0$, $v \in V$ and $\tilde{\phi} \in L^\infty\left([0,T]\times\Omega, \mathrm{d}t\otimes P;\mathbb{R}\right)$. For $\phi = \bar{X} - \varepsilon\tilde{\phi}v$ in (2.3.33) we get

$$
\begin{aligned}
0 \overset{(2.3.33)}{\geqslant} &\mathbb{E}\Bigg[\int_0^T \psi_t\Bigg(\int_0^{\hat{t}} e^{-\int_0^s (F_r+\varrho(\bar{X}_r-\varepsilon\tilde{\phi}_r v))\mathrm{d}r}\Big(2\,{}_{V^*}\Big\langle Y_s - A\left(s,\bar{X}_s - \varepsilon\tilde{\phi}_s v\right), \varepsilon\tilde{\phi}_s v\Big\rangle_V\\
&\quad - \left(F_s + \varrho\left(\bar{X}_s - \varepsilon\tilde{\phi}_s v\right)\right)\left\|\varepsilon\tilde{\phi}_s v\right\|_H^2\Big)\,\mathrm{d}s\Bigg)\,\mathrm{d}t\Bigg]\\
=\varepsilon\,&\mathbb{E}\Bigg[\int_0^T \psi_t\Bigg(\int_0^{\hat{t}} e^{-\int_0^s (F_r+\varrho(\bar{X}_r-\varepsilon\tilde{\phi}_r v))\mathrm{d}r}\Big(2\tilde{\phi}_s\,{}_{V^*}\Big\langle Y_s - A\left(s,\bar{X}_s - \varepsilon\tilde{\phi}_s v\right), v\Big\rangle_V\\
&\quad - \varepsilon\left(F_s + \varrho\left(\bar{X}_s - \varepsilon\tilde{\phi}_s v\right)\right)\left\|\tilde{\phi}_s v\right\|_H^2\Big)\,\mathrm{d}s\Bigg)\,\mathrm{d}t\Bigg]
\end{aligned}
\tag{2.3.34}
$$

By the hemicontinuity (A1) the map $\varepsilon \mapsto {}_{V^*}\Big\langle Y_s - A\left(s,\bar{X}_s - \varepsilon\tilde{\phi}_s v\right), v\Big\rangle_V$ is continuous, hence we have $\lim_{\varepsilon\to 0} {}_{V^*}\Big\langle Y_s - A\left(s,\bar{X}_s - \varepsilon\tilde{\phi}_s v\right), v\Big\rangle_V = {}_{V^*}\langle Y_s - A\left(s,\bar{X}_s\right), v\rangle_V$. Since we assumed ϱ to be hemicontinuous, too, we also conclude $\lim_{\varepsilon\to 0}\varrho\left(\bar{X}_s - \varepsilon\tilde{\phi}_s v\right) = \varrho\left(\bar{X}_s\right)$. Now, by dividing by $\varepsilon\,(>0)$ and then letting $\varepsilon \to 0$ in (2.3.34) we come to

$$
0 \geqslant \mathbb{E}\left[\int_0^T \psi_t\left(\int_0^{\hat{t}} e^{-\int_0^s (F_r+\varrho(\bar{X}_r))\mathrm{d}r}\left(2\tilde{\phi}_s\,{}_{V^*}\langle Y_s - A\left(s,\bar{X}_s\right), v\rangle_V\right)\mathrm{d}s\right)\mathrm{d}t\right].
$$

Again, by the arbitrariness of ψ and $\tilde{\phi}$ and with $R \to \infty$ we can finally conclude

$$Y = A\left(\cdot, \bar{X}\right) \quad \mathrm{d}t \otimes P\text{-a.e.}$$

□

We still lack the proof of the regularity estimates 2.2.1 (i) and 2.2.1 (ii).

2.3.13 Corollary. *Let $(X_t)_{t\in[0,T]}$ the stochastic process defined in (2.3.23).*

(i) For the $\mathrm{d}t \otimes P$-version $\bar{X}$ of X we have

$$\bar{X} \in L^{\alpha}\left([0,T] \times \Omega, \mathrm{d}t \otimes P; V\right) \cap L^2\left([0,T] \times \Omega, \mathrm{d}t \otimes P; H\right).$$

(ii) There exists a constant $\tilde{C} = \tilde{C}\left(p, \gamma, \theta, C, K, T\right) > 0$ such that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X(t)\|_H^p\right] \leqslant \tilde{C}\left(\mathbb{E}\left[\|X_0\|_H^p\right] + \mathbb{E}\left[\int_0^T F_t^{\frac{p}{2}}\,\mathrm{d}t\right] + 1\right).$$

(iii) If $0 \leqslant \gamma < \Gamma$, then for the $\mathrm{d}t \otimes P$-version $\bar{X}$ of X we have

$$\bar{X} \in L^p\left(\Omega; L^{\infty}\left([0,T]; H\right)\right)$$

and there exists a constant $\hat{C} = \hat{C}\left(p, \gamma, \theta, C, C_{BDG}, K, T\right) > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|_H^p\right] \leqslant \hat{C}\left(\mathbb{E}\left[\|X_0\|_H^p\right] + \mathbb{E}\left[\int_0^T F_t^{\frac{p}{2}}\,\mathrm{d}t\right] + 1\right).$$

Proof. **Part (i).** We note that $\bar{X} \in L^{\alpha}\left([0,T] \times \Omega, \mathrm{d}t \otimes P; V\right) = \mathfrak{L}^{\alpha}$ is already fulfilled by Lemma 2.3.5 (i). Since $\bar{X} = X$ $\mathrm{d}t \otimes P$-a.e. by Lemma 2.3.9 we deduce from Proposition 2.3.10 (ii)

$$\mathbb{E}\left[\int_0^T \|\bar{X}_t\|_H^2\,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T \|X_t\|_H^2\,\mathrm{d}t\right] \leqslant \mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^2 \int_0^T \mathrm{d}t\right]$$
$$= T\,\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^2\right] < \infty,$$

therefore $\bar{X} \in L^2\left([0,T] \times \Omega, \mathrm{d}t \otimes P; H\right)$.

Part (ii). By Proposition 2.3.10 (i), $(X_t)_{t\in[0,T]}$ is càdlàg and hence $t \mapsto \|X_t\|_H^p$ is right lower semicontinuous and therefore also $t \mapsto \mathbb{E}\left[\|X_t\|_H^p\right]$. By Lemma 2.3.7 (i), the $\mathrm{d}t \otimes P$-version $\bar{X}$ of X is also the weakly star limit of $\left(X^{(n_k)}\right)_{k\in\mathbb{N}}$ in $L^{\infty}\left([0,T], \mathrm{d}t; L^p\left(\Omega; H\right)\right)$. Hence we have together with Lemma E.2

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X_t\|_H^p\right] = \operatorname*{ess\,sup}_{t\in[0,T]} \mathbb{E}\left[\|X_t\|_H^p\right] = \operatorname*{ess\,sup}_{t\in[0,T]} \mathbb{E}\left[\|\bar{X}_t\|_H^p\right]$$
$$\leqslant \sup_{t\in[0,T]} \mathbb{E}\left[\|\bar{X}_t\|_H^p\right] \leqslant \liminf_{k\to\infty}\left(\sup_{t\in[0,T]} \left\|X_t^{(n_k)}\right\|_H^p\right)$$
$$\overset{(2.3.4)}{\leqslant} \tilde{C}\left(\mathbb{E}\left[\|X_0\|_H^p\right] + \mathbb{E}\left[\int_0^T F_t^{\frac{p}{2}}\,\mathrm{d}t\right] + 1\right),$$

where $\tilde{C} = \tilde{C}(p, \gamma, \theta, C, K, T) > 0$.
Part (iii). Since $0 \leqslant \gamma < \Gamma$, we can use Lemma 2.3.5 (iii) to see that $X^{(n_k)} \to \bar{X}$ weakly star in $L^p([0,T]; L^\infty(\Omega; H))$ for the $\mathrm{d}t \otimes P$-version $\bar{X}$ of X. By Proposition 2.3.10 (i), $(X_t)_{t\in[0,T]}$ is càdlàg and hence $t \mapsto \|X_t\|_H^p$ is right lower semicontinuous. Therefore Lemma E.2 and (2.3.5) yields to

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^p\right] &= \mathbb{E}\left[\operatorname*{ess\,sup}_{t\in[0,T]} \|X_t\|_H^p\right] = \mathbb{E}\left[\operatorname*{ess\,sup}_{t\in[0,T]} \left\|\bar{X}_t\right\|_H^p\right] \\
&\leqslant \mathbb{E}\left[\sup_{t\in[0,T]} \left\|\bar{X}_t\right\|_H^p\right] \leqslant \liminf_{k\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \left\|X^{(n_k)}\right\|_H^p\right] \\
&\overset{(2.3.5)}{\leqslant} \hat{C}\left(\mathbb{E}\left[\|X_0\|_H^p\right] + \mathbb{E}\left[\int_0^T F_t^{\frac{p}{2}}\,\mathrm{d}t\right] + 1\right),
\end{aligned}$$

where $\hat{C} = \hat{C}(p, \gamma, \theta, C, C_{BDG}, K, T) > 0$.

□

2.4. Proof of the Main Theorem – Uniqueness

2.4.1 Proposition. *Let $(X_t)_{t\in[0,T]}$, $(Y_t)_{t\in[0,T]}$ be solutions of 2.1.1 with initial values $X_0, Y_0 \in L^{\beta+2}(\Omega, \mathcal{F}_0, P; H)$ respectively given by Theorem 2.2.1, such that $X_0 = Y_0$ P-a.s. Let $0 \leqslant \gamma < \Gamma$. Then P-a.s.*

$$X_t = Y_t$$

for every $t \in [0,T]$.

2.4.2 Remark. *The two solutions are pathwise unique by the path càdlàg property of X and Y in H.*

Proof. We are P-a.s. in the following situation for $t \in [0,T]$:

$$\begin{aligned}
X_t &= X_0 + \int_0^t A(s, X_s)\,\mathrm{d}s + \int_0^t B(s, X_s)\,\mathrm{d}W_s + \int_0^t \int_Z f(s, X_{s-}, z)\,\bar{\mu}(\mathrm{d}s, \mathrm{d}z), \\
Y_t &= Y_0 + \int_0^t A(s, Y_s)\,\mathrm{d}s + \int_0^t B(s, Y_s)\,\mathrm{d}W_s + \int_0^t \int_Z f(s, Y_{s-}, z)\,\bar{\mu}(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

Let $\bar{Y}$ the $\mathrm{d}t \otimes P$-version of Y. We apply Itô's formula from 2.3.10 (i) together with the product rule and P-a.s. come to

$$\begin{aligned}
&e^{-\int_0^t (K+\varrho(\bar{Y}_s))\mathrm{d}s} \|X_t - Y_t\|_H^2 \\
=&\|X_0 - Y_0\|_H^2 + \int_0^t e^{-\int_0^s (K+\varrho(\bar{Y}_r))\mathrm{d}r} \Big(2\,{}_{V^*}\langle A(s,X_s) - A(s,Y_s), X_s - Y_s\rangle_V \\
&+ \|B(s,X_s)\|_{L_2}^2 - \left(K + \varrho(\bar{Y}_s)\right) \|X_s - Y_s\|_H^2 \Big) \,\mathrm{d}s \\
&+ \int_0^t \int_Z e^{-\int_0^s (K+\varrho(\bar{Y}_r))\mathrm{d}r} \|f(s,X_{s-},z) - f(s,Y_{s-},z)\|_H^2 \,\mu(\mathrm{d}s,\mathrm{d}z) \\
&+ 2\int_0^t e^{-\int_0^s (K+\varrho(\bar{Y}_r))\mathrm{d}r} \langle X_s - Y_s, B(s,X_s)\,\mathrm{d}W_s - B(s,Y_s)\,\mathrm{d}W_s\rangle_H \\
&+ 2\int_0^t \int_Z e^{-\int_0^s (K+\varrho(\bar{Y}_r))\mathrm{d}r} \langle X_s - Y_s, f(s,X_{s-},z) - f(s,Y_{s-},z)\rangle_H \,\bar{\mu}(\mathrm{d}s,\mathrm{d}z)\,.
\end{aligned}$$

We apply expectation to both sides and use Proposition 1.2.7. Then the local monotonicity condition (A2) leads to

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_0^t (K+\varrho(\bar{Y}_s))\mathrm{d}s} \|X_t - Y_t\|_H^2\right] \leqslant \mathbb{E}\left[\|X_0 - Y_0\|_H^2\right] \\
&+ 2\,\mathbb{E}\left[\int_0^t e^{-\int_0^s (K+\varrho(\bar{Y}_r))\mathrm{d}r} \langle X_s - Y_s, B(s,X_s)\,\mathrm{d}W_s - B(s,Y_s)\,\mathrm{d}W_s\rangle_H\right] \\
&+ 2\,\mathbb{E}\left[\int_0^t \int_Z e^{-\int_0^s (K+\varrho(\bar{Y}_r))\mathrm{d}r} \langle X_s - Y_s, f(s,X_{s-},z) - f(s,Y_{s-},z)\rangle_H \,\bar{\mu}(\mathrm{d}s,\mathrm{d}z)\right].
\end{aligned}$$

By Corollary 2.3.11 the expectation of the appearing martingales is zero. Therefore we conclude

$$\mathbb{E}\left[e^{-\int_0^t (K+\varrho(\bar{Y}_s))\mathrm{d}s} \|X_t - Y_t\|_H^2\right] \leqslant \mathbb{E}\left[\|X_0 - Y_0\|_H^2\right]$$

and by the assumption that $X_0 = Y_0$ P-a.s.

$$0 \leqslant \mathbb{E}\left[e^{-\int_0^t (K+\varrho(\bar{Y}_s))\mathrm{d}s} \|X_t - Y_t\|_H^2\right] \leqslant 0 \quad \text{for all } t \in [0,T]\,.$$

If we can show that $\int_0^T \left(K + \varrho(\bar{Y}_s)\right)\,\mathrm{d}s < \infty$ P-a.s., then we are done. By condition (B3)

$$\begin{aligned}
&\int_0^T \left(K + \varrho(\bar{Y}_s)\right)\,\mathrm{d}s \leqslant C\int_0^T \left(1 + \|\bar{Y}_s\|_V^\alpha\right)\left(1 + \|\bar{Y}_s\|_H^\beta\right)\,\mathrm{d}s \\
=&C\left[T + \int_0^T \|\bar{Y}_s\|_V^\alpha\,\mathrm{d}s + \int_0^T \|\bar{Y}_s\|_H^\beta\,\mathrm{d}s + \int_0^T \|\bar{Y}_s\|_V^\alpha \|\bar{Y}_s\|_H^\beta\,\mathrm{d}s\right] \\
\leqslant&C\left[T + \int_0^T \|\bar{Y}_s\|_V^\alpha\,\mathrm{d}s + \sup_{s\in[0,T]} \|\bar{Y}_s\|_H^\beta \left(T + \int_0^T \|\bar{Y}_s\|_V^\alpha\,\mathrm{d}s\right)\right] \\
\leqslant&C\left[\left(1 + \frac{2}{\beta+2} + \frac{\beta}{\beta+2}\sup_{s\in[0,T]}\left(\|\bar{Y}_s\|_H^{\beta+2}\right)\right)\left(T + \int_0^T \|\bar{Y}_s\|_V^\alpha\,\mathrm{d}s\right)\right],
\end{aligned} \tag{2.4.1}$$

where we used Youngs's inequality with $q = \frac{\beta+2}{\beta}$ in the last step. This step is unnecessary if $\beta = 0$, cf. Remark 2.3.6 (i). The right hand side of (2.4.1) is P-a.e. finite, since $\bar{Y} \in L^\alpha\left([0,T] \times \Omega, \mathrm{d}t \otimes P; V\right)$ and $\bar{Y} \in L^p\left(\Omega; L^\infty\left([0,T]; H\right)\right)$ by Lemma 2.3.13 (i) and 2.3.13 (iii) respectively.

□

3. Application to Examples

In this chapter we are going to establish existence and uniqueness results to semi-linear and quasi-linear stochastic partial differential equations. Our main references are [LR10], [PR07], [BLZ11] and [LS14].

Notation. *We denote the i-th spatial derivative $\frac{\partial}{\partial x_i}$ by D_i. For $q > 1$ let q' its dual such that $\frac{1}{q} + \frac{1}{q'} = 1$. If a Hilbert space H_1 is continuously embedded into another Hilbert space H_2, we denote this with $H_1 \hookrightarrow H_2$.*

Let $H_0^{1,p}(\Lambda; \mathbb{R})$ denote the standard Sobolev space with the Sobolev norm

$$\|u\|_V = \|u\|_{H_0^{1,p}} := \left(\int_\Lambda |\nabla u(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}, \quad u \in H_0^{1,p}(\Lambda; \mathbb{R}).$$

We set $L^p(\Lambda) := L^p(\Lambda; \mathbb{R}) =: H_0^{0,p}(\Lambda)$ and $H_0^{1,p}(\Lambda; \mathbb{R}) =: H_0^{1,p}(\Lambda)$.
Let $\Lambda \subset \mathbb{R}^d$, $d \in \mathbb{N}$, an open, bounded domain and let ξ denote the Lebesgue measure on Λ. For $1 \leqslant p < \infty$ we consider the Gelfand triple

$$V := H_0^{1,p}(\Lambda; \mathbb{R}) \subset H := L^2(\Lambda; \mathbb{R}) \subset \left(H_0^{1,p}(\Lambda; \mathbb{R}) \right)^* = V^*,$$

Let $0 < T < \infty$ fixed. Let $(\Omega, \mathcal{F}, P)$ be a probability space with normal filtration $(\mathcal{F}_t)_t$, $t \geqslant 0$. Let $(Z, \mathcal{Z}, m)$ a measurable space with a σ-finite measure m on it. Fix a stationary $(\mathcal{F}_t)$-Poisson point processes $\mathfrak{p}$ on Z and $(\Omega, \mathcal{F}, P)$ and let $\bar{\mu}$ the compensated Poisson random measure to $\mathfrak{p}$. Let $(W_t)_{t \geqslant 0}$ a U-valued cylindrical Wiener process on $(\Omega, \mathcal{F}_t, P)$, where U is a separable Hilbert space.

3.1. Semilinear stochastic equations

Let $p = 2$. Then our Gelfand triple has the form

$$V = H_0^{1,2}(\Lambda) \subset L^2(\Lambda) = H \subset H_0^{-1,2}(\Lambda) = V^*.$$

For $1 \leqslant i \leqslant d$ let $b_i, f_i \colon \Lambda \to \mathbb{R}$ and set $b := (b_1, \ldots, b_d)$, $f := (f_1, \ldots, f_d)$. For $u \in V = H_0^{1,2}(\Lambda)$ we define the operators

$$A_f(u) := \Delta u + \langle f(u), \nabla u \rangle_{\mathbb{R}^d},$$
$$A_b(u) := \Delta u + \langle b, \nabla u \rangle_{\mathbb{R}^d}$$

and

$$A_{f,f_0}(u) := A_f(u) + f_0(u) \ \Big(= \Delta u + \sum_{i=1}^{d} f_i(u) D_i u + f_0(u) \Big),$$
$$A_{b,f_0}(u) := A_b(u) + f_0(u) \ \Big(= \Delta u + \sum_{i=1}^{d} b_i D_i u + f_0(u) \Big).$$

Consider the following equation

$$\begin{aligned} \mathrm{d}X(t) &= A(X(t))\,\mathrm{d}t + B(X(t))\,\mathrm{d}W(t) + \int_Z g(X(t-), z)\,\bar{\mu}(\mathrm{d}z, \mathrm{d}s), \\ X(0) &= X_0, \end{aligned} \tag{3.1.1}$$

where $A := \begin{cases} A_{f,f_0}, & \text{if } d = 1 \text{ or } d = 2, \\ A_{b,f_0}, & \text{else} \end{cases}$ and W and $\bar{\mu}$ fulfill the same properties as in Section 2.1. Suppose that there exist $C_{SL}, r, s, \eta \geqslant 0$ such that the following holds:

(SL1) $f = (f_1, \ldots, f_d) : \mathbb{R} \to \mathbb{R}^d$ is Lipschitz with Lipschitz constant L_f.

(SL2) $f_0 \colon \mathbb{R} \to \mathbb{R}$ is continuous with $f_0(0) = 0$ and satisfies for all $x, y \in \mathbb{R}$

$$|f_0(x)| \leqslant C_{SL}(1 + |x|^r),$$
$$(f_0(x) - f_0(y))(x - y) \leqslant C_{SL}(1 + |y|^s)(x - y)^2.$$

(SL3) $B \colon H_0^{1,2}(\Lambda; \mathbb{R}) \to L_2\big(U; L^2(\Lambda; \mathbb{R})\big)$ satisfies for all $v_1, v_2 \in H_0^{1,2}(\Lambda; \mathbb{R})$

$$\|B(v_1) - B(v_2)\|_{L_2}^2 \leqslant C_{SL}\left(1 + \int_\Lambda |\nabla v_2|^2\,\mathrm{d}\xi\right) \int_\Lambda |v_1 - v_2|^2\,\mathrm{d}\xi.$$

(SL4) $g \colon \mathbb{R} \times Z \to \mathbb{R}$ such that for all $v, v_1, v_2 \in H_0^{1,2}(\Lambda; \mathbb{R})$ we have

$$\int_Z \int_\Lambda |g(v_1, z) - g(v_2, z)|^2\,\mathrm{d}\xi\, m(\mathrm{d}z) \leqslant C_{SL}\left(1 + \int_\Lambda |\nabla v_2|^2\,\mathrm{d}\xi\right) \int_\Lambda |v_1 - v_2|^2\,\mathrm{d}\xi,$$
$$\begin{aligned} \int_Z \left(\int_\Lambda |g(v, z)|^2\,\mathrm{d}\xi\right)^\zeta m(\mathrm{d}z) \leqslant C_{SL} &\left(1 + \left(\int_\Lambda |v|^2\,\mathrm{d}\xi\right)^\zeta\right) \\ &+ \eta \left(\int_\Lambda |v|^2\,\mathrm{d}\xi\right)^{\zeta - 1} \left(\int_\Lambda |\nabla v|^2\,\mathrm{d}\xi\right) \end{aligned}$$

where $\zeta = \begin{cases} 3, & \text{if } d = 1, \\ \max\{2; r\}, & \text{if } d = 2, \\ \frac{8}{3}, & \text{else.} \end{cases}$

3.1.1 Remark. *(i) The Lipschitz condition on f implies for all $x \in \mathbb{R}$*

$$|f(x)| \leqslant |f(x) - f(0)| + |f(0)| \leqslant L_f |x| + |f(0)|.$$

(ii) The weakened Lipschitz condition on B in (SL3) implies linear growth with respect to $\|\cdot\|_H$ restricted to V: By (SL3) we have for all $v \in H_0^{1,2}(\Lambda)$

$$\begin{aligned}\|B(v)\|_{L_2} &= \|B(v) - B(0) + B(0)\|_{L_2} \leqslant \|B(v) - B(0)\|_{L_2} + \|B(0)\|_{L_2} \\ &\leqslant \sqrt{C_{SL}}\, \|v\|_H + \|B(0)\|_{L_2}.\end{aligned}$$

Set $L_0 := \|B(0)\|_{L^2}$. Then

$$\|B(v)\|_{L_2}^2 \leqslant C_{SL} \|v\|_H^2 + L_0^2 + 2\sqrt{C_{SL}} L_0 \|v\|_H.$$

Young's inequality gives

$$2\left(\sqrt{C_{SL}} L_0 \|v\|_H\right) \leqslant 2\left(\frac{1}{2} C_{SL} L_0^2 + \frac{1}{2}\|v\|_H^2\right),$$

and we conclude

$$\|B(v)\|_{L_2}^2 \leqslant (C_{SL} + 1)\|v\|_H^2 + (C_{SL} + 1) L_0^2.$$

Hence, with $L_B := (1 + C_{SL})\left(1 + L_0^2\right) > 0$, we have for all $v \in H_0^{1,2}(\Lambda)$

$$\|B(v)\|_{L_2}^2 \leqslant L_B\left(1 + \|v\|_H^2\right).$$

(iii) We have the same inequality for g. Define $\mathrm{Int}_m(\varphi) := \int_Z \varphi(z)\, m(\mathrm{d}z)$. Then for all $u \in H_0^{1,2}(\Lambda)$ by Young's inequality and condition (SL4) we have for $L_0 := \mathrm{Int}_m\left(\|g(0)\|_H^2\right)$

$$\begin{aligned}\mathrm{Int}_m\left(\|g(u)\|_H^2\right) &\leqslant \mathrm{Int}_m\left(\left(\|g(u) - g(0)\|_H + \|g(0)\|_H\right)^2\right) \\ &\leqslant \mathrm{Int}_m\left(\|g(u) - g(0)\|_H^2\right) + L_0 + 2\,\mathrm{Int}_m\left(\|g(u) - g(0)\|_H \|g(0)\|_H\right) \\ &\overset{Young}{\leqslant} 2\,\mathrm{Int}_m\left(\|g(u) - g(0)\|_H^2\right) + 2L_0 \leqslant 2C_{SL}\|u\|_H^2 + 2L_0,\end{aligned}$$

hence for $L_g := 2(C_{SL} + L_0)$

$$\begin{aligned}\int_Z \|g(v,z)\|_H^2\, m(\mathrm{d}z) &= \int_Z \int_\Lambda |g(v,z)|^2\, \mathrm{d}x\, m(\mathrm{d}z) \leqslant L_g\left(1 + \int_\Lambda |v|^2\, \mathrm{d}x\right) \\ &= L_g\left(1 + \|v\|_H^2\right).\end{aligned}$$

(iv) *If B is assumed to be Lipschitz from $H_0^{1,2}(\Lambda) \to L_2\left(U; L^2(\Lambda;\mathbb{R})\right)$ with respect to $\|\cdot\|_H$ restricted to V as in [BLZ11, Example 4.3] or [LR10, Example 3.2], then B fulfills (SL3).*

Here in our examples, also condition (SL4) has been weakened in the same way contrary to [BLZ11, Example 4.3] and we allow $\int_Z \left(\int_\Lambda |g(v,z)|^2 \,\mathrm{d}\xi\right)^\zeta m(\mathrm{d}z)$ also to be bounded by another term.

It is stated in [BLZ11, Remark 4.4 (3)] that one can weaken (SL3) even to

$$\begin{aligned}\|B(v_1) - B(v_2)\|_{L_2}^2 &\leqslant \int_\Lambda |\nabla(v_1 - v_2)|^2 \,\mathrm{d}x + C_{SL}\left(1 + \int_\Lambda |\nabla v_2|^2 \,\mathrm{d}x\right)\int_\Lambda |v_1 - v_2|^2 \,\mathrm{d}x \\ &= \|v_1 - v_2\|_V^2 + C_{SL}\left(1 + \|v_2\|_V^2\right)\|v_1 - v_2\|_H^2 ,\end{aligned}$$

and by a similiar argument as in (ii) one can get

$$\|B(v)\|_{L_2}^2 \leqslant \bar{L}\left(1 + \|v\|_V^2 + \|v\|_H^2\right)$$

for an $\bar{L} > 0$. But this condition must be connected to a further assumption on B, e.g. in the coercivity condition (A3) in the following examples, we can handle $\bar{L}\|v\|_V^2$ only if $\bar{L} < 1$, because Lemma 3.1.2 provides $-1 \cdot \|v\|_V^2$. Since we need that $\theta > 0$, we would need that $1 - \bar{L} > 0$.

Another problem would be the appearing of the $\|v_1 - v_2\|_V^2$-term in the local monotonicity condition (A2), which we cannot handle there, since ϱ only affects the second variable v_2. A solution is to substitute $\|v_1 - v_2\|_V^2$ with $\|v_2\|_V^2$, but this case is already covered by our condition (SL3).

3.1.2 Lemma. *(i) Suppose (SL1) holds and let $d \in \{1;2;3\}$. There exists a constant $C_1 > 0$ such that for all $u, v, w \in V = H_0^{1,2}(\Lambda, \mathbb{R})$ we have*

$$\int_\Lambda |u|\,|\nabla v|\,|w| \,\mathrm{d}\xi \leqslant C_1 \|u\|_V \|v\|_V \|w\|_V .$$

(ii) *Suppose (SL1) holds. If $d \in \{1;2\}$, then there exists a constant $C_2 \geqslant 0$ such that for all $u, v \in V$*

$$2\,{}_{V^*}\langle A_f(u) - A_f(v), u - v\rangle_V \leqslant -\|u - v\|_V^2 + C_2\left(1 + \|v\|_V^2\right)\|u - v\|_H^2 .$$

(iii) *Let $d = 3$ and $b_i \in L^d(\Lambda) + L^\infty(\Lambda)$ for $1 \leqslant i \leqslant d$. Then there exists a constant $C_3 > 0$ such that for all $u, v \in V$ we have*

$$2\,{}_{V^*}\langle A_b(u) - A_b(v), u - v\rangle_V \leqslant -\|u - v\|_V^2 + C_3\|u - v\|_H^2 .$$

Proof. This proof is a slightly modification from [LR14].
Part (i): By Hölder's inequality for $q = 2 = q'$ we have

$$\int_\Lambda |u|\,|\nabla v|\,|w| \,\mathrm{d}\xi \leqslant \|v\|_V \|uw\|_{L^2(\Lambda)} . \tag{3.1.2}$$

So it is still to show that $\|uw\|_{L^2(\Lambda)} \leqslant C_1 \|u\|_V \|w\|_V$ for a constant $C_1 > 0$.
Let $d = 1$. Then by Theorem F.1 (i) and Proposition F.3 there exist constants $C_{2,V}, C_{\infty,V} > 0$ such that $V \hookrightarrow L^2(\Lambda)$ and $V \hookrightarrow L^\infty(\Lambda)$ respectively. Hence with $C_1 := C_{2,V} C_{\infty,V}$ we get

$$\|uw\|_{L^2(\Lambda)} \leqslant \|u\|_{L^\infty(\Lambda)} \|w\|_{L^2(\Lambda)} \leqslant C_1 \|u\|_V \|w\|_V .$$

Let $d = 2$. By Theorem F.1 (i) there exists a constant $C_{4,V} > 0$ such that $V \hookrightarrow L^4(\Lambda)$. Then for $C_1 := C_{4,V}^2$ and Hölder's inequality with $q = 2 = q'$

$$\|uw\|_{L^2(\Lambda)} \overset{\text{Hölder}}{\leqslant} \|u\|_{L^4(\Lambda)} \|w\|_{L^4(\Lambda)} \leqslant C_1 \|u\|_V \|w\|_V .$$

Let $d = 3$. There exist constants $C_{6,V} > 0$, $C_{3,V} > 0$ such that $V \hookrightarrow L^6(\Lambda)$ and $V \hookrightarrow L^3(\Lambda)$ respectively by Theorem F.1 (i), since $1 \leqslant k \leqslant \frac{2d}{d-2} = 6$ for $k \in \{3, 6\}$. For $C_1 := C_{6,V} C_{3,V}$ and by Hölder's inequality with $q = 3$, $q' = \frac{3}{2}$ we get

$$\|uv\|_{L^2(\Lambda)} \overset{\text{Hölder}}{\leqslant} \|u\|_{L^6(\Lambda)} \|w\|_{L^3(\Lambda)} \leqslant C_1 \|u\|_V \|w\|_V .$$

Part (ii): Let $u, v \in V$. Integration by parts gives ${}_{V^*}\langle \Delta(u-v), u-v \rangle_V = -\|u-v\|_V^2$. For $i \in \{1, \ldots, d\}$ let $F_i, G_i \colon \mathbb{R} \to \mathbb{R}$ such that $F_i(0) = 0 = G_i(0)$ and $D_i F_i = f_i$, $D_i G_i = F_i$. Using that $u = 0 = v$ on $\partial\Lambda$ since $u, v \in H_0^{1,2}(\Lambda) = V$, we have $G_i(u-v) = 0$ on $\partial\Lambda$ and we get by the chainrule, integration by parts and Gauss' divergence theorem

$$\begin{aligned} &{}_{V^*}\langle \langle f(u-v), \nabla u - v \rangle_{\mathbb{R}^d}, u-v \rangle_V = \sum_{i=1}^d \int_\Lambda f_i(u-v) D_i(u-v)(u-v)\, \mathrm{d}\xi \\ \overset{\substack{\text{Chain-}\\ \text{rule}}}{=} &\sum_{i=1}^d \int_\Lambda D_i(F_i(u-v))(u-v)\, \mathrm{d}\xi \overset{\substack{\text{Int. by}\\ \text{parts}}}{=} -\sum_{i=1}^d \int_\Lambda D_i(G_i(u-v))\, \mathrm{d}\xi \overset{\text{Gauss}}{=} 0. \end{aligned} \tag{3.1.3}$$

This leads to

$$\begin{aligned} &{}_{V^*}\langle \langle f(u), \nabla u \rangle_{\mathbb{R}^d} - \langle f(v), \nabla v \rangle_{\mathbb{R}^d}, u-v \rangle_V \\ =& {}_{V^*}\langle \langle f(u), \nabla(u-v) \rangle_{\mathbb{R}^d} + \langle f(u) - f(v), \nabla v \rangle_{\mathbb{R}^d}, u-v \rangle_V \\ \overset{(3.1.3)}{=}& {}_{V^*}\langle \langle f(u) - f(u-v), \nabla(u-v) \rangle_{\mathbb{R}^d} + \langle f(u) - f(v), \nabla v \rangle_{\mathbb{R}^d}, u-v \rangle_V \end{aligned} \tag{3.1.4}$$

Let $d = 1$. By Proposition F.3 there exists a constant $C_{\infty,V} > 0$ for the continuous embedding $V \hookrightarrow L^\infty(\Lambda)$. Together with the Lipschitz continuity of f and Hölder's inequality for $q = 2 = q'$ the right hand side of (3.1.4) is bounded by

$$\begin{aligned} &L_f \int_\Lambda |v| |\nabla(u-v)| |u-v|\, \mathrm{d}\xi + L_f \int_\Lambda |u-v| |\nabla v| |u-v|\, \mathrm{d}\xi. \\ \leqslant& L_f \|v\|_\infty \|u-v\|_V \|u-v\|_H + L_f \|u-v\|_\infty \|v\|_V \|u-v\|_H \\ \leqslant& L_f C_{\infty,V} \|v\|_V \|u-v\|_V \|u-v\|_H + L_f C_{\infty,V} \|u-v\|_V \|v\|_V \|u-v\|_H \\ \leqslant& \|u-v\|_V \|u-v\|_H \cdot \bar{C}_1 (1 + \|v\|_V), \end{aligned}$$

where $\bar{C}_1 := 2L_f C_{\infty,V}$.
Let $d = 2$ and f_1, f_2 bounded. With Hölder's inequality for $q = 2 = q'$ and Lemma F.5 (i) the right hand side of (3.1.4) can be estimated by

$$\begin{aligned}
&\|f\|_\infty \|u-v\|_V \|u-v\|_H + L_f \|v\|_V \|u-v\|_{L^4(\Lambda)}^2 \\
\leqslant &\|f\|_\infty \|u-v\|_V \|u-v\|_H + 2L_f \|v\|_V \|u-v\|_V \|u-v\|_H \\
\leqslant &\|u-v\|_V \|u-v\|_H \cdot \bar{C}_2 (1 + \|v\|_V),
\end{aligned}$$

where $\bar{C}_2 := \|f\|_\infty + 2L_f$.
Now let $d \in \{1; 2\}$. Then by the estimates above, Young's inequality for $q = 2 = q'$ and Lemma B.3 we get

$$\begin{aligned}
&2\,{}_{V^*}\langle A_f(u) - A_f(v), u-v\rangle_V = 2\,{}_{V^*}\langle \Delta(u-v) + \langle f(u), \nabla u\rangle_{\mathbb{R}^d} - \langle f(v), \nabla v\rangle_{\mathbb{R}^d}, u-v\rangle_V \\
\leqslant & -2\|u-v\|_V^2 + 2\cdot \|u-v\|_V \cdot \|u-v\|_H \cdot \bar{C}_d (1 + \|v\|_V) \\
\overset{Young}{\underset{B.3}{\leqslant}} & -\|u-v\|_V^2 + \frac{1}{2}\cdot 2^{2-1} \cdot \bar{C}_d \|u-v\|_H^2 \left(1 + \|v\|_V^2\right).
\end{aligned}$$

Therefore the claim follows by choosing $C_2 = \bar{C}_d$.
Part (iii): Let $u, v \in V$ and let $C_3 > 0$ the constant from Lemma F.6 in case $\varepsilon = \frac{1}{2}$. Then by integrating by parts and using that A_b is linear, we get

$$\begin{aligned}
&2\,{}_{V^*}\langle A_b(u) - A_b(v), u-v\rangle_V = 2\int_\Lambda (\Delta(u-v))(u-v)\,\mathrm{d}\xi + 2\int_\Lambda \langle\langle b, \nabla(u-v)\rangle_{\mathbb{R}^d}, u-v\rangle\,\mathrm{d}\xi \\
\leqslant & -2\int_\Lambda |\nabla(u-v)|\,|\nabla(u-v)|\,\mathrm{d}\xi + 2\int_\Lambda |b|\,|\nabla(u-v)|\,|u-v|\,\mathrm{d}\xi \\
\overset{F.6}{\leqslant} & -2\|u-v\|_V^2 + 2\left(\frac{1}{2}\|u-v\|_V^2 + C_3\|u-v\|_H^2\right) = -\|u-v\|_V^2 + C_3\|u-v\|_H^2.
\end{aligned}$$

□

3.1.1. Examples

3.1.3 Example ($d = 1$)**.** Suppose (SL1) to (SL4) hold for $d = 1$, $r = 3$, $s = 2$ and $\eta < \frac{1}{106}$. Then for any initial value $X_0 \in L^{\bar{p}}\left(\Omega, \mathcal{F}_0, P; L^2(\Lambda, \mathbb{R})\right)$, where $\bar{p} \geqslant 6$, equation (3.1.1) with operator $A = A_{f,f_0}$ has a solution $X = (X_t)_{t\in[0,T]}$ which fulfills

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X(t)\|_H^6\right] < \infty.$$

For $\eta = 0$ this solution is unique and we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|_H^6\right] < \infty.$$

Proof. This proof is divided into claims to verify conditions (A1)–(A4) and (B1)–(B3) and to show that $A\colon V \to V^*$ for $A = A_{f,f_0}$. Finally, we will use these claims to show that there exists a solution and that this solution is unique. In the situation of Theorem 2.2.1 we will verify the needed conditions for

$$\alpha = 2, \quad \beta = 4, \quad \gamma = \eta,$$
$$\theta = 1, \quad K := C_0, \quad C := C_0$$

and $F :\equiv C_0$, $\varrho(v) := C_0 \left(1 + \|v\|_V^2\right)\left(1 + \|v\|_H^4\right)$ for $v \in V$, where $C_0 > 0$ is a constant big enough. In the following proof we will see how big C_0 has to be.
Claim: $\gamma < \theta \frac{\beta+2}{2} \cdot \left[(\beta+2)(\beta+1) + 2^{\beta+1}(2\beta+1)\right]^{-1}$.
Since $\theta = 1$ and $\beta = 4$ we calculate

$$\theta \frac{\beta+2}{2} \cdot \left[(\beta+2)(\beta+1) + 2^{\beta+1}(2\beta+1)\right]^{-1} = \frac{3}{6 \cdot 5 + 9 \cdot 2^5} = \frac{1}{106} > \eta = \gamma.$$

Claim: $A_{f,f_0}\colon V \to V^*$.
Let $u, v \in V$, then by (SL2) with $r = 3$

$$\begin{aligned}
&\left|{}_{V^*}\langle f_0(u), v\rangle_V\right| \leqslant \int_\Lambda |f_0(u)|\,|v|\,\mathrm{d}x \\
\leqslant &C_{SL}\int_\Lambda \left(1 + |u|^3\right)|v|\,\mathrm{d}x = C_{SL}\int_\Lambda \left(|v| + |u|^3|v|\right)\mathrm{d}x \leqslant C_{SL}\int_\Lambda \left(|v| + \sup_\Lambda |v| \cdot \sup_\Lambda |u| \cdot |u|^2\right)\mathrm{d}x \\
= &C_{SL}\left(\|v\|_{L^1(\Lambda)} + \|v\|_{L^\infty(\Lambda)}\|u\|_{L^\infty(\Lambda)}\|u\|_H^2\right) \\
\leqslant &C_{SL}\left(C_{1,V}\|v\|_V + C_{\infty,V}^2\|v\|_V\|u\|_V\|u\|_H^2\right) \\
\leqslant &\tilde{C}\|v\|_V\left(1 + \|u\|_V\|u\|_H^2\right),
\end{aligned}$$

where $C_{1,V}$ is the constant from the continuous embedding $V = H_0^{1,2}(\Lambda) \hookrightarrow L^1(\Lambda)$ (cf. Theorem F.1), $C_{\infty,V} > 0$ the constant from the embedding $V \hookrightarrow L^\infty(\Lambda)$ (cf. Proposition F.3) and $\tilde{C} = C_{SL}\left(C_{1,V} + C_{\infty,V}^2\right)$. Furthermore we have

$$\left|{}_{V^*}\langle \Delta u, v\rangle_V\right| \leqslant \|u\|_V\|v\|_V$$

by Lemma A.3. By Theorem F.1 (i) we have $V \hookrightarrow L^2(\Lambda)$ with constant $C_{2,V} > 0$ and by Proposition F.3 $V \hookrightarrow L^\infty(\Lambda)$ with constant $C_{\infty,V} > 0$. Since $f = f_1$ is Lipschitz by condition (SL1), with $\bar{C} = |f(0)| \cdot C_{2,V} + L_f \cdot C_{\infty,V}$ we conclude by Hölder's inquality and

Lemma B.3

$$\left|{}_{V^*}\langle\langle f(u),\nabla u\rangle_{\mathbb{R}},v\rangle_V\right| \leqslant \int_\Lambda |f(u)|\,|\nabla u|\,|v|\,\mathrm{d}\xi \overset{\text{(SL1)}}{\leqslant} \int_\Lambda (|f(0)|+L_f|u|)\,|\nabla u|\,|v|\,\mathrm{d}\xi$$
$$\overset{\text{Hölder}}{\leqslant} |f(0)|\,\|u\|_V\,\|v\|_{L^2(\Lambda)} + L_f\,\|v\|_\infty\,\|u\|_V\,\|u\|_H$$
$$\leqslant |f(0)|\,\|u\|_V\cdot C_{2,V}\,\|v\|_V + L_f\cdot C_{\infty,V}\,\|v\|_V\,\|u\|_V\,(1+\|u\|_H)$$
$$\leqslant \bar{C}\,\|u\|_V\,\|v\|_V + \bar{C}\,\|v\|_V\,\|u\|_V\,(1+\|u\|_H)^2$$
$$\overset{B.3}{\leqslant} \bar{C}\,\|u\|_V\,\|v\|_V + 2\bar{C}\,\|v\|_V\,\|u\|_V\left(1+\|u\|_H^2\right).$$

Hence we have

$$\left|{}_{V^*}\langle A_{f,f_0}(u),v\rangle_V\right| \leqslant \|u\|_V\,\|v\|_V + \tilde{C}\,\|v\|_V\left(1+\|u\|_V\,\|u\|_H^2\right)$$
$$+\,2\bar{C}\left(\|u\|_V\,\|v\|_V + \|v\|_V\,\|u\|_V\left(1+\|u\|_H^2\right)\right)$$
$$= \|v\|_V\left(\tilde{C} + \|u\|_V\left(1+4\bar{C}+\left(\tilde{C}+2\bar{C}\right)\|u\|_H^2\right)\right). \tag{3.1.5}$$

and we conclude $A_{f,f_0}(u)\in V^*$.
Claim: (A1) holds.
Let $u,v,w\in V$ and $\lambda\in\mathbb{R}$ with $|\lambda|\leqslant 1$. We have to show

$$0 = \lim_{\lambda\to 0}\left({}_{V^*}\langle A_{f,f_0}(u+\lambda v),w\rangle_V - {}_{V^*}\langle A_{f,f_0}(u),w\rangle_V\right).$$
$$= \lim_{\lambda\to 0}\left(\int_\Lambda\Big(\langle\Delta(u+\lambda v) + \langle f(u+\lambda v),\nabla(u+\lambda v)\rangle_{\mathbb{R}} + f_0(u+\lambda v),w\rangle\right.$$
$$\left. - \langle\Delta u + \langle f(u),\nabla u\rangle_{\mathbb{R}} + f_0(u),w\rangle\Big)\,\mathrm{d}\xi\right).$$

Since Δ is linear, f_0 is assumed to be continuous from condition (SL2) and f is continuous, too (since it is assumed to be Lipschitz in condition (SL1)), we have convergence to zero for the integrands dξ-a.e. The claim then follows by Lebesgue's dominated convergence theorem, because by $|\lambda|\leqslant 1$ and conditions (SL1) and (SL2) we have

$$\langle\Delta(u+\lambda v) + \langle f(u+\lambda v),\nabla(u+\lambda v)\rangle_{\mathbb{R}} + f_0(u+\lambda v),w\rangle$$
$$= \langle\Delta u+\lambda\Delta v,w\rangle + \langle\langle f(u+\lambda v),\nabla(u+\lambda v)\rangle_{\mathbb{R}},w\rangle + \langle f_0(u+\lambda v),w\rangle$$
$$\leqslant \langle\Delta u,w\rangle + \langle\Delta v,w\rangle + \langle|f(u+\lambda v)|\,|\nabla u|,w\rangle + \langle|f(u+\lambda v)|\,|\nabla v|,w\rangle + |f_0(u+\lambda v)|\,|w|$$
$$\underset{\text{(SL2)}}{\overset{\text{(SL1)}}{\leqslant}} \langle\Delta u,w\rangle + \langle\Delta v,w\rangle + (|f(0)|+L_f(|u|+|v|))\,|\nabla u|\,|w|$$
$$+\,(|f(0)|+L_f(|u|+|v|))\,|\nabla v|\,|w| + C_{SL}\,(1+(|u|+|v|)^r)\,|w|. \tag{3.1.6}$$

This term dominates the integrands and it remains to show that it is integrable.

$$\int_\Lambda |w|\,\mathrm{d}\xi = \|w\|_{L^1(\Lambda)} \leqslant C_{1,V}\,\|w\|_V < \infty,$$

because $V \hookrightarrow L^1(\Lambda)$ by Theorem F.1 (i) with constant $C_{1,V} > 0$. With Hölder's inequality for $q = 2 = q'$ we get

$$\int_\Lambda \langle \Delta u, w\rangle \, \mathrm{d}\xi = -\int_\Lambda \langle \nabla u, \nabla v\rangle \, \mathrm{d}\xi \overset{\text{Hölder}}{\leqslant} \left| \|u\|_V \|v\|_V \right| < \infty.$$

By Lemma 3.1.2 (i) we have

$$\int_\Lambda |v| \, |\nabla u| \, |w| \, \mathrm{d}\xi \leqslant C_1 \|u\|_V \|v\|_V \|w\|_V < \infty.$$

Finally, since $r = 3$, by Lemma B.3, Hölder's inequality for $q = 2 = q'$ and Theorem F.1 (i)

$$\begin{aligned} \int_\Lambda (|u| + |v|)^r \, |w| \, \mathrm{d}\xi &\overset{B.3}{\leqslant} 4 \int_\Lambda \left(|u|^3 + |v|^3 \right) |w| \, \mathrm{d}\xi \overset{\text{Hölder}}{\leqslant} 4 \|w\|_{L^2(\Lambda)} \left(\|u\|_{L^6(\Lambda)}^3 + \|v\|_{L^6(\Lambda)}^3 \right) \\ &\overset{F.1\,(i)}{\leqslant} 4 C_{2,V} \|w\|_V C_{6,V}^3 \left(\|u\|_V^3 + \|v\|_V^3 \right) \\ &< \infty, \end{aligned}$$

where $C_{2,V} > 0$ and $C_{6,V}$ are the constants from the embeddings $V \hookrightarrow L^2(\Lambda)$ and $V \hookrightarrow L^6(\Lambda)$ respectively. For completeness we also note

$$\int_\Lambda |\nabla u| \, |w| \, \mathrm{d}\xi \overset{\text{Hölder}}{\leqslant} \|u\|_V \|w\|_{L^2(\Lambda)} \leqslant C_{2,V} \|u\|_V \|w\|_V < \infty.$$

Claim: (A2) holds.
By (SL2) we have for all $u, v \in H_0^{1,2}(\Lambda; \mathbb{R})$

$$\begin{aligned} 2\,{}_{V^*}\langle f_0(u) - f_0(v), u - v\rangle_V &= 2 \int_\Lambda (f_0(u) - f_0(v))(u - v) \, \mathrm{d}\xi \\ &\overset{(SL2)}{\leqslant} C_{SL} \int_\Lambda (1 + |v|^s)(u - v)^2 \, \mathrm{d}\xi \leqslant C_{SL} \int_\Lambda \left(1 + \sup_\Lambda |v|^s \right) (u - v)^2 \, \mathrm{d}\xi \\ &= C_{SL} \left(1 + \|v\|_{L^\infty(\Lambda)}^s \right) \|u - v\|_{L^2(\Lambda)}^2 . \end{aligned}$$

We have $\|v\|_{L^\infty(\Lambda)} \leqslant C_{\infty,V} \|v\|_V$, where $C_{\infty,V} > 0$ is the constant from the continuous embedding $V = H_0^{1,2}(\Lambda) \hookrightarrow L^\infty(\Lambda)$. Therefore, since $s = 2$,

$$2\,{}_{V^*}\langle f_0(u) - f_0(v), u - v\rangle_V \leqslant C_{SL} \left(1 + C_{\infty,V} \|v\|_V^2 \right) \|u - v\|_H^2 .$$

By Lemma 3.1.2 (ii) we have for all $u, v \in V$

$$2\,{}_{V^*}\langle A_f(u) - A_f(v), u - v\rangle_V \leqslant C_2 \left(1 + \|v\|_V^2 \right) \|u - v\|_H^2 .$$

Now, by (SL3), for all $u, v \in V$

$$\|B(u) - B(v)\|_{L_2}^2 \leqslant C_{SL} \left(1 + \int_\Lambda |\nabla v|^2 \, \mathrm{d}\xi \right) \int_\Lambda |u - v|^2 \, \mathrm{d}x = C_{SL} \left(1 + \|v\|_V^2 \right) \|u - v\|_H^2$$

and by (SL4)

$$\begin{aligned}&\int_Z \|g(u,z)-g(v,z)\|_H^2 \, m(\mathrm{d}z) \leqslant \int_Z \int_\Lambda |g(u,z)-g(v,z)|^2 \, \mathrm{d}\xi \, m(\mathrm{d}z) \\ \leqslant &C_{SL} \left(1+\int_\Lambda |\nabla v|^2 \, \mathrm{d}\xi\right) \int_\Lambda |u-v|^2 \, \mathrm{d}\xi = C_{SL}\left(1+\|v\|_V^2\right) \|u-v\|_H^2 .\end{aligned}$$

Altogether we see that (A2) is fulfilled:

$$\begin{aligned}&2 \, {}_{V^*}\langle A_{f,f_0}(u) - A_{f,f_0}(v), u-v\rangle_V + \|B(u)-B(v)\|_{L_2}^2 + \int_Z \|g(u,z)-g(v,z)\|_H^2 \, m(\mathrm{d}z) \\ \leqslant &(C_2 + C_{SL}(3+C_{\infty,V}))\left(1+\|v\|_V^2\right)\|u-v\|_H^2 \\ \leqslant &\varrho(v)\|u-v\|_H^2 \leqslant (F_t+\varrho(v))\|u-v\|_H^2 ,\end{aligned}$$

for all $t \in [0,T]$ since F is non-negative and $C_0 \geqslant C_2 + C_{SL}(3+C_{\infty,V})$.
Claim: (A3) holds.
Let $u \in V$. Then by Lemma 3.1.2 (ii)

$$\begin{aligned}&2 \, {}_{V^*}\langle A_f(u), u\rangle_V + \|u\|_V^2 = 2 \, {}_{V^*}\langle A_f(u) - A_f(0), u-0\rangle_V + \|u-0\|_V^2 \\ \leqslant &C_2 \|u\|_H^2 \leqslant C_2\left(1+\|u\|_H^2\right)\end{aligned}$$

and we have by 3.1.1 (ii)

$$\|B(u)\|_{L_2}^2 \leqslant L_B\left(1+\|u\|_H^2\right).$$

Since $f_0(0)=0$ we have by (SL2)

$$\begin{aligned}&2 \, {}_{V^*}\langle f_0(u), u\rangle_V = 2 \, {}_{V^*}\langle f_0(u) - f_0(0), u-0\rangle_V \\ \leqslant &2C_{SL}\int_\Lambda u^2 \, \mathrm{d}\xi \leqslant 2C\|u\|_H^2 \leqslant 2C_{SL}\left(1+\|u\|_H^2\right).\end{aligned}$$

Summarizing yields to

$$\begin{aligned}&2 \, {}_{V^*}\langle A_{f,f_0}(u), u\rangle_V + \|B(u)\|_{L_2}^2 + \|u\|_V^2 \\ \leqslant &(C_2+L_B+2C_{SL})\left(1+\|u\|_H^2\right) = F_t + K\|u\|_H^2 ,\end{aligned}$$

for all $t \in [0,T]$, where $F :\equiv C_0 =: K$ and $C_0 \geqslant C_2 + L_b + 2C_{SL}$.
Claim: (A4) holds.
Let $u \in V$. For $\alpha = 2$ and $\beta = 4$ we calculate for the operator norm of A_{f,f_0} by (3.1.5)

with $\hat{C} = \left(2\tilde{C} + 6\bar{C}\right)^2$ and Lemma B.3

$$\begin{aligned}
\|A_{f,f_0}(u)\|_{V^*}^2 =& \left(\sup_{\substack{v\in V,\\ \|v\|_V=1}} \left|{}_{V^*}\langle A_{f,f_0}(u), v\rangle_V\right| \right)^2 \\
\overset{(3.1.5)}{\leqslant}& \left(\tilde{C} + \|u\|_V \left(1 + 4\bar{C} + \left(\tilde{C} + 2\bar{C}\right)\|u\|_H^2\right)\right)^2 \\
\leqslant& \hat{C}\left(1 + \|u\|_V \left(1 + \|u\|_H^2\right)\right)^2 \\
\overset{B.3}{\leqslant}& 2\hat{C}\left(1 + \|u\|_V^2 \left(1 + \|u\|_H^2\right)^2\right) \overset{B.3}{\leqslant} 2\hat{C}\left(1 + 2\|u\|_V^2 \left(1 + \|u\|_H^4\right)\right) \\
\leqslant& 4\hat{C}\left(1 + \|u\|_V^2 + \|u\|_V^2 \|u\|_H^4\right) \\
\leqslant& \left(F_t + K\|u\|_V^2\right)\left(1 + \|u\|_H^4\right),
\end{aligned}$$

where $F :\equiv C_0 =: K$ and $C_0 \geqslant 4\hat{C}$.
Claim: (B1) holds.
By Remark 3.1.1 (ii) and (iii) we have for all $u \in V$

$$\begin{aligned}
&\|B(u)\|_{L_2}^2 + \int_Z \|g(u,z)\|_H^2 \, m(\mathrm{d}z) \leqslant (L_B + L_g)\left(1 + \|u\|_H^2\right) \\
\leqslant& C\left(1 + F_t + \|u\|_H^2\right),
\end{aligned}$$

for all $t \in [0,T]$ where $C := C_0 \equiv: F_t$ and $C_0 \geqslant L_B + L_g$.
Claim: (B2) holds.
Let $u \in V$. Since $\zeta = 3 = \frac{6}{2} = \frac{\beta+2}{2}$ we get by (SL4)

$$\begin{aligned}
&\int_Z \|g(u,z)\|_H^{\beta+2} \, m(\mathrm{d}z) = \int_Z \left(\int_\Lambda |g(u,z)|^2 \, \mathrm{d}\xi\right)^\zeta m(\mathrm{d}z) \\
\leqslant& C_{SL}\left(1 + \left(\int_\Lambda |u|^2 \, \mathrm{d}\xi\right)^\zeta\right) + \eta \left(\int_\Lambda |v|^2 \, \mathrm{d}\xi\right)^{\zeta-1} \left(\int_\Lambda |\nabla v|^2 \, \mathrm{d}\xi\right) \\
=& C_{SL}\left(1 + \|u\|_H^{\beta+2}\right) + \eta \|v\|_H^{2(\zeta-1)} \|v\|_V^2 \\
\leqslant& C\left(1 + F_t^{\frac{\beta+2}{2}} + \|u\|_H^{\beta+2}\right) + \gamma \|v\|_H^\beta \|v\|_V^\alpha,
\end{aligned} \tag{3.1.7}$$

for all $t \in [0,T]$, where $C := C_0 \equiv: F$ and $C_0 \geqslant C_{SL}$.
Claim: (B3) holds.
Since $\alpha = 2$, $\beta = 4$ and $C := C_0$ we have for $u \in V$

$$\varrho(u) = C_0\left(1 + \|u\|_V^2\right)\left(1 + \|u\|_H^4\right) = C\left(1 + \|u\|_V^\alpha\right)\left(1 + \|u\|_H^\beta\right).$$

Claim: (3.1.1) with $A = A_{f,f_0}$ has a solution.
With the previous claims we can apply Theorem 2.2.1 (i) and get a solution $X = (X_t)_{t\in[0,T]}$. Furthermore we have

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X(t)\|_H^6\right] < \infty.$$

Claim: If $\eta = 0$, then the solution is unique.
Suppose $\eta = \gamma = 0$, then we obtain uniqueness by Theorem 2.2.1 (ii) and

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|_H^6\right] < \infty.$$

□

3.1.4 Example ($d = 2$)**.** Suppose (SL1) to (SL4) hold for $d = 2$, $1 < r < 3$, $s = 2$ and for $\eta < \begin{cases} \frac{1}{40}, & \text{if } r \leqslant 2, \\ \left[4r + 2^{2r+1}\right]^{-1}, & \text{else.} \end{cases}$ For $i \in \{1,2\}$ let f_i be bounded. Then for any $\bar{p} \geqslant \max\{2r; 4\}$ and initial value $X_0 \in L^{\bar{p}}\left(\Omega, \mathcal{F}_0, P; L^2(\Lambda, \mathbb{R})\right)$ equation (3.1.1) with operator $A = A_{f,f_0}$ has a solution $X = (X_t)_{t\in[0,T]}$ which fulfills

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X(t)\|_H^{\max\{2r;4\}}\right] < \infty.$$

If $\eta = 0$, then this solution is unique and we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|_H^{\max\{2r;4\}}\right] < \infty.$$

Proof. The structure of this proof is identical to the proof of Example 3.1.3. We will verify Theorem 2.2.1 for

$$\begin{aligned} &\alpha = 2, \quad \beta = \max\{2(r-1); 2\}, \quad \gamma = \eta \\ &\theta = 1, \quad K := C_0, \qquad\qquad\qquad C := C_0 \end{aligned}$$

and $F :\equiv C_0$, $\varrho(v) = C_0\left(1 + \|v\|_V^2\right)\left(1 + \|v\|_H^\beta\right)$, where the constant $C_0 > 0$ is big enough, which we will see in the following proof.
Claim: $\gamma < \theta\frac{\beta+2}{2} \cdot \left[(\beta+2)(\beta+1) + 2^{\beta+1}(2\beta+1)\right]^{-1}$.
With Remark 2.2.2 we see that the claim follows, since

$$\gamma = \eta < \frac{1}{2}\left[\left(\max\{2r;4\} + 2^{\max\{2r;4\}}\right)\right]^{-1} = \frac{\theta}{2}\left[(\beta+2) + 2^{\beta+2}\right]^{-1}.$$

Claim: $A_{f,f_0} : V \to V^*$.
Note that since $p = 2 = d$ we have $V \hookrightarrow L^q(\Lambda)$ for all $q \geqslant 1$ by Theorem F.1 (i). Let $u, v \in V$ and $\varepsilon \in (0,1)$. Then by (SL2) since $f_0(0) = 0$ and Hölder's inequality with

$q = \varepsilon + 1$, $q' = \frac{\varepsilon+1}{\varepsilon}$

$$\begin{aligned}
\left| {}_{V^*}\langle f_0(u), v\rangle_V \right| &\leqslant C_{SL} \int_\Lambda (1 + |u|^r)|v| \, \mathrm{d}x \\
&\leqslant C_{SL} \left(\|v\|_{L^1(\Lambda)} + \left(\int_\Lambda |u|^{r(\varepsilon+1)} \, \mathrm{d}x \right)^{\frac{1}{\varepsilon+1}} \left(\int_\Lambda |v|^{\frac{\varepsilon+1}{\varepsilon}} \, \mathrm{d}x \right)^{\frac{\varepsilon}{\varepsilon+1}} \right) \\
&= C_{SL} \left(\|v\|_{L^1(\Lambda)} + \|u\|^r_{L^{r(\varepsilon+1)}(\Lambda)} \|v\|_{L^{\frac{\varepsilon+1}{\varepsilon}}(\Lambda)} \right) \\
&\leqslant C_{SL} \left(C_{1,V} \|v\|_V + C_{\frac{\varepsilon+1}{\varepsilon},V} \|v\|_V \|u\|^r_{L^{r(\varepsilon+1)}} \right),
\end{aligned}$$

where the constants $C_{1,V}$, $C_{\frac{\varepsilon+1}{\varepsilon},V} > 0$ are from the continuous embeddings $V \hookrightarrow L^1(\Lambda)$ and $V \hookrightarrow L^{\frac{\varepsilon+1}{\varepsilon}}(\Lambda)$. Since $r < 3$, we can choose ε so small that $\varepsilon < \frac{3-r}{r-1}$. This implies $r < 1 + \frac{2}{\varepsilon+1}$. By Hölder's inequality and with $\lambda \in (0,1)$ arbitrary we get

$$\begin{aligned}
\|u\|^r_{L^{r(\varepsilon+1)}} &= \left(\int_\Lambda |u|^{\lambda r(\varepsilon+1)} |u|^{(1-\lambda)r(\varepsilon+1)} \, \mathrm{d}\xi \right)^{\frac{1}{\varepsilon+1}} \\
&\leqslant \left(\int_\Lambda |u|^{\lambda r(\varepsilon+1)q} \, \mathrm{d}\xi \right)^{\frac{1}{q(\varepsilon+1)}} \left(\int_\Lambda |u|^{(1-\lambda)r(\varepsilon+1)q'} \right)^{\frac{1}{q'(\varepsilon+1)}} \\
&= \|u\|^{\lambda r}_{L^{\lambda r q(\varepsilon+1)}(\Lambda)} \|u\|^{(1-\lambda)r}_{L^{(1-\lambda)rq'(\varepsilon+1)}(\Lambda)}.
\end{aligned}$$

Therefore, by choosing $\lambda = \frac{1}{r}$ and $q = \frac{2}{2-(r-1)(\varepsilon+1)}$, $q' = \frac{2}{(r-1)(\varepsilon+1)}$ we get

$$\|u\|^r_{L^{r(\varepsilon+1)}} \leqslant \|u\|_{L^{(\varepsilon+1)q}(\Lambda)} \|u\|^{r-1}_{L^2(\Lambda)} \leqslant C_{q,V} \|u\|_V \|u\|^{r-1}_H,$$

where we used Theorem F.1 (i) in the last step. Hence

$$\left| {}_{V^*}\langle f_0(u), v\rangle_V \right| \leqslant \tilde{C} \|v\|_V \left(1 + \|u\|_V \|u\|^{r-1}_H \right),$$

where $\tilde{C} = C_{SL} \cdot \max \left\{ C_{1,V}; C_{\frac{\varepsilon+1}{\varepsilon},V} C_{q,V} \right\}$. By Lemma A.3 we have

$$\left| {}_{V^*}\langle \Delta u, v\rangle_V \right| \leqslant \|u\|_V \|v\|_V.$$

Since f is bounded, we get with Hölder's inequality for $q = 2 = q'$, Lemma 3.1.2 (i) and the continuous embedding $V \hookrightarrow L^2(\Lambda)$ with constant $C_{2,V} > 0$

$$\begin{aligned}
\left| {}_{V^*}\langle \langle f(u), \nabla u\rangle_{\mathbb{R}^2}, v\rangle_V \right| &\leqslant \int_\Lambda |f(u)| |\nabla u| |v| \, \mathrm{d}\xi \\
&\overset{\text{Hölder}}{\leqslant} \|f\|_\infty \|u\|_V \|v\|_{L^2(\Lambda)} \\
&\overset{3.1.2\,(i)}{\leqslant} C_{2,V} \|f\|_\infty \|u\|_V \|v\|_V.
\end{aligned}$$

Hence by Lemma A.2 and B.3

$$
\begin{aligned}
\left|{}_{V^*}\langle A_{f,f_0}(u), v\rangle_V\right| \leqslant{}& \tilde{C}\,\|v\|_V\left(1+\|u\|_V\,\|u\|_H^{r-1}\right)+\|u\|_V\,\|v\|_V \\
&+C_{2,V}\,\|f\|_\infty\,\|u\|_V\,\|v\|_V \\
\leqslant{}& \bar{C}\,\|v\|_V\left(1+\|u\|_V+\|u\|_V\,\|u\|_H^{r-1}\right)
\end{aligned} \tag{3.1.8}
$$

where $\bar{C}=1+\tilde{C}+(1+C_{2,V}\,\|f\|_\infty)$ and we conclude $A_{f,f_0}(u)\in V^*$.
Claim: (A1) holds.
Let $u,v,w\in V$, $\lambda\in\mathbb{R}$ with $|\lambda|<1$. As in the proof of condition (A1) in Example 3.1.3 we only have to show that (3.1.6) is integrable. To do this we note that since we have $V\hookrightarrow L^q(\Lambda)$ for all $q\geqslant 1$ by Theorem F.1 (i), all the embeddings used there also work in case $d=2$, except for

$$
\int_\Lambda (|u|+|v|)^r\,|w|\,\mathrm{d}\xi.
$$

By Hölder's inequality with $q=2=q'$ we have

$$
\int_\Lambda |u|\,|w|\,\mathrm{d}\xi\leqslant \|u\|_{L^2(\Lambda)}\,\|w\|_{L^2(\Lambda)}\leqslant C_{2,V}^2\,\|u\|_V\,\|w\|_V<\infty,
$$

where $C_{2,V}>0$ is the constant from the embedding $V\hookrightarrow L^2(\Lambda)$. Again, Hölder's inequality with $q=\frac{r+1}{r}$ and $q'=r+1$ we get

$$
\begin{aligned}
\int_\Lambda |v|^r\,|w|\,\mathrm{d}\xi &\leqslant \left(\int_\Lambda |v|^{r+1}\,\mathrm{d}\xi\right)^{\frac{r}{r+1}}\left(\int_\Lambda |w|^{r+1}\,\mathrm{d}\xi\right)^{\frac{1}{r+1}}=\|v\|_{L^{r+1}(\Lambda)}^r\,\|w\|_{L^{r+1}(\Lambda)} \\
&\leqslant C_{r+1,V}^{r+1}\,\|v\|_V^r\,\|w\|_V\,,
\end{aligned}
$$

where $C_{r+1,V}>0$ is the constant from the continuous embedding $V\hookrightarrow L^{r+1}(\Lambda)$ by Theorem F.1 (i).
Claim: (A2) holds.
Let $u,v\in V$. By (SL2) with $s=2$, Hölder's inequality with $q=2$, Lemma F.5 (i) and Young's inequality we get

$$
\begin{aligned}
{}_{V^*}\langle f_0(u)-f_0(v), u-v\rangle_V &\leqslant C_{SL}\int_\Lambda (1+|v|^s)\,(u-v)^2\,\mathrm{d}x \\
&\leqslant C_{SL}\,\|u-v\|_H^2+C_{SL}\int_\Lambda |v|^2\,|u-v|^2\,\mathrm{d}x \\
&\overset{\text{Hölder}}{\leqslant} C_{SL}\,\|u-v\|_H^2+C_{SL}\left(\int_\Lambda |v|^{2\cdot 2}\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Lambda |u-v|^{2\cdot 2}\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&= C_{SL}\,\|u-v\|_H^2+C_{SL}\,\|v\|_{L^4(\Lambda)}^2\,\|u-v\|_{L^4(\Lambda)}^2 \\
&\overset{F.5\,(i)}{\leqslant} C_{SL}\,\|u-v\|_H^2+4C_{SL}\,\|u\|_V\,\|u\|_H\,\|u-v\|_H\,\|u-v\|_V \\
&\overset{\text{Young}}{\leqslant} C_{SL}\,\|u-v\|_H^2+8C_{SL}^2\,\|u\|_V^2\,\|u\|_H^2\,\|u-v\|_H^2+\frac{1}{2}\,\|u-v\|_V^2\,.
\end{aligned}
$$

Now by 3.1.2 (ii), since f_1, f_2 are bounded and $d = 2$,

$$2\,{}_{V^*}\langle A_f(u) - A_f(v), u - v\rangle_V \overset{3.1.2\,(ii)}{\leqslant} C_2\left(1 + \|v\|_V^2\right)\|u - v\|_H^2 - \|u - v\|_V^2 .$$

By (SL3) we get

$$\|B(u) - B(v)\|_{L_2}^2 \leqslant C_{SL}\int_\Lambda |u - v|^2\,\mathrm{d}x = C_{SL}\left(1 + \|v\|_V^2\right)\|u - v\|_H^2$$

and by (SL4)

$$\int_Z \|g(u,z) - g(v,z)\|_H^2\,m(\mathrm{d}z) \leqslant C_{SL}\int_\Lambda |u - v|^2\,\mathrm{d}x = C_{SL}\left(1 + \|v\|_V^2\right)\|u - v\|_H^2 .$$

Combining these results leads to (A2):

$$\begin{aligned}
&2\,{}_{V^*}\langle A_{f,f_0}(u) - A_{f,f_0}(v), u - v\rangle_V + \|B(u) - B(v)\|_{L_2}^2 + \int_Z \|g(u,z) - g(v,z)\|_H^2\,m(\mathrm{d}z)\\
\leqslant&\,\|u - v\|_V^2 - \|u - v\|_V^2 + \left(8C_{SL}^2\|u\|_V^2\|u\|_H^2 + (2C_{SL} + C_2)\|u\|_V^2\right)\|u - v\|_H^2\\
\leqslant&\,\hat{C}\left(\|u\|_V^2\|u\|_H^2 + \|u\|_V^2 + \|u\|_H^2 + 1\right)\|u - v\|_H^2\\
=&\,\hat{C}\left(1 + \|u\|_V^2\right)\left(1 + \|u\|_H^2\right)\|u - v\|_H^2 \overset{A.2}{\leqslant} C_0\left(1 + \|u\|_V^2\right)\left(1 + \|u\|_H^\beta\right)\|u - v\|_H^2\\
=&\,\varrho(v)\|u - v\|_H^2 ,
\end{aligned}$$

with $\hat{C} := \max\left\{C_2 + 4C_{SL}; 8C_{SL}^2\right\}$ and $C_0 \geqslant 2^{\frac{\beta}{2}-1}\hat{C}$.
Claim: (A3) holds.
Let $u \in V$. Lemma 3.1.2 (ii) gives

$$2\,{}_{V^*}\langle A_f(u), u\rangle_V \leqslant -\|u\|_V^2 + C_2\|u\|_H^2 = -\theta\|u\|_V^\alpha + C_2\|u\|_H^2 .$$

By Remark 3.1.1 (ii) we have

$$\|B(u)\|_{L_2}^2 \leqslant L_B\left(1 + \|u\|_H^2\right) = L_B + L_B\|u\|_H^2 .$$

Therefore, since $f_0(0) = 0$, with condition (SL2)

$$\begin{aligned}
&2\,{}_{V^*}\langle A_{f,f_0}(u), u\rangle_V + \|B(u)\|_{L_2}^2 + \theta\|u\|_V^\alpha \leqslant (C_2 + L_B)\|u\|_H^2 + 2C_{SL}\int_\Lambda u^2\,\mathrm{d}\xi + L_B\\
\leqslant&\,L_B + (C_2 + L_B + 2C_{SL})\|u\|_H^2 \leqslant F_t + K\|u\|_H^2
\end{aligned}$$

for all $t \in [0,T]$, where $C_0 \geqslant C_2 + L_B + 2C_{SL}$. (Remember $K = C_0$, $F \equiv C_0$.)

Claim: (A4) holds.
Let $u \in V$. For $\alpha = 2$, the operator norm of A_{f,f_0} can be estimated by (3.1.8) and Lemma B.3

$$\|A_{f,f_0}(u)\|_{V^*}^{\frac{\alpha}{\alpha-1}} = \left(\sup_{\substack{v \in V, \\ \|v\|_V = 1}} \left| {}_{V^*}\langle A_{f,f_0}(u), v \rangle_V \right| \right)^2$$

$$\overset{(3.1.8)}{\leqslant} \left(\bar{C} \left(1 + \|u\|_V + \|u\|_V \|u\|_H^{r-1} \right) \right)^2 \overset{B.3}{\leqslant} 2\bar{C}^2 \left(1 + \|u\|_V^2 \left(1 + \|u\|_H^{r-1} \right)^2 \right)$$

$$\overset{B.3}{\leqslant} 4\bar{C}^2 \left(1 + \|u\|_V^2 \left(1 + \|u\|_H^{2(r-1)} \right) \right) \overset{A.2}{\leqslant} 4\bar{C}^2 \left(1 + 2^{\frac{\beta}{2(r-1)}-1} \|u\|_V^2 \left(1 + \|u\|_H^{\beta} \right) \right)$$

$$\leqslant F_t + K \|u\|_V^{\alpha} + K \|u\|_V^{\alpha} \|u\|_H^{\beta} + F_t \|u\|_H^{\beta} = (F_t + K\|v\|_V^{\alpha}) \left(1 + \|v\|_H^{\beta} \right)$$

for all $t \in [0,T]$, where $F :\equiv C_0 =: K$ and $C_0 \geqslant 2^{\frac{\beta}{2(r-1)}+1}\bar{C}^2$.
Claim: (B1) holds.
This is exactely the same proof as in Example 3.1.3.
Claim: (B2) holds.
This follows from (3.1.7), since $\zeta = \max\{2; r\} = \frac{\max\{2r;4\}}{2} = \frac{\max\{2(r-1);2\}+2}{2} = \frac{\beta+2}{2}$.
Claim: (B3) holds.
The definition of ϱ already fulfills (B3).
Claim: (3.1.1) with $A = A_{f,f_0}$ has a solution.
Theorem 2.2.1 (i) applied to (3.1.1) with $A = A_{f,f_0}$ gives us a solution $X = (X_t)_{t\in[0,T]}$ and we have

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X(t)\|_H^{\max\{2r;4\}} \right] < \infty,$$

since $\beta + 2 = \max\{2r; 4\}$.
Claim: If $\eta = 0$, then the solution is unique.
Suppose $\eta = \gamma = 0$, then Theorem 2.2.1 (ii) provides the uniqueness of the solution and we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|_H^{\max\{2r;4\}} \right] < \infty.$$

□

3.1.5 Example ($d = 3$)**.** Suppose (SL2) to (SL4) hold for $d = 3$, $r = \frac{7}{3}$, $s = \frac{4}{3}$ and $\eta < \left[2\left(\frac{16}{3} + 32\sqrt[3]{2}\right)\right]^{-1}$. For $i \in \{1,2,3\}$ let $b_i \in L^d(\Lambda) + L^\infty(\Lambda)$. Then for any $\bar{p} \geqslant \frac{16}{3}$ and initial value $X_0 \in L^{\bar{p}}\left(\Omega, \mathcal{F}_0, P; L^2(\Lambda, \mathbb{R})\right)$ equation (3.1.1) with operator $A = A_{b,f_0}$ has a solution $X = (X_t)_{t\in[0,T]}$ which fulfills

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X(t)\|_H^{\frac{16}{3}} \right] < \infty.$$

If $\eta = 0$, then this solution is unique and we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|_H^{\frac{16}{3}}\right] < \infty.$$

Proof. The structure of this proof is identical to the proof of Example 3.1.3, but with $A = A_{b,f_0}$. We will verify Theorem 2.2.1 for

$$\alpha = 2, \quad \beta = \frac{10}{3}, \quad \gamma = \eta$$
$$\theta = 1, \quad K := C_0, \quad C := C_0$$

and $F :\equiv C_0$, $\varrho(v) = C_0\left(1 + \|v\|_V^2\right)\left(1 + \|v\|_H^{\frac{10}{3}}\right)$, where the constant $C_0 > 0$ is big enough, which we will see in the following proof.

Claim: $\gamma < \theta^{\frac{\beta+2}{2}} \cdot \left[(\beta+2)(\beta+1) + 2^{\beta+1}(2\beta+1)\right]^{-1}$.

$\gamma = \eta$ has been chosen in a way such that the claim follows from Remark 2.2.2, since

$$\gamma = \eta < \left[2\left(\frac{16}{3} + 32\sqrt[3]{2}\right)\right]^{-1} = \frac{\theta}{2}\left[(\beta+2) + 2^{\beta+2}\right]^{-1}.$$

Claim: $A_{b,f_0} \colon V \to V^*$.

Let $u, v \in V$. Condition (SL2) with Hölder's inequality for $q = \frac{2d}{d+2}\ \left(= \frac{6}{5}\right)$, $q' = \frac{2d}{d-2}\ (= 6)$ gives

$$\left|{}_{V^*}\langle f_0(u), v\rangle_V\right| \leqslant C_{SL}\int_\Lambda (1 + |u|^r)\,|v|\ \mathrm{d}x$$
$$= C_{SL}\left(\|v\|_{L^1(\Lambda)} + \int_\Lambda |u|^r\,|v|\ \mathrm{d}x\right) \overset{\text{Hölder}}{\leqslant} C_{SL}\left(\|v\|_{L^1(\Lambda)} + \left(\int_\Lambda (|u|^r)^q\right)^{\frac{1}{q}}\left(\int_\Lambda |v|^{q'}\right)^{\frac{1}{q'}}\right)$$
$$= C_{SL}\left(\|v\|_{L^1(\Lambda)} + \|u\|^r_{L^{\frac{2d}{d+2}}(\Lambda)}\|v\|_{L^{\frac{2d}{d-2}}(\Lambda)}\right).$$

By the continuous Sobolev embedding $H_0^{1,2}(\Lambda) \hookrightarrow L^q(\Lambda)$ for all $1 \leqslant q \leqslant \frac{dp}{d-p} = 6$ from Theorem F.1 (i) there exist constants $C_{6,V} > 0$ and $C_{1,V} > 0$ such that we have $\|v\|_{L^{\frac{2d}{d-2}}(\Lambda)} \leqslant C_{6,V}\|v\|_V$ and $\|v\|_{L^1(\Lambda)} \leqslant C_{1,V}\|v\|_V$. Hence, for $\tilde{C} = \max\{C_{1,V}; C_{6,V}; 1\}$ we have

$$\left|{}_{V^*}\langle f_0(u), v\rangle_V\right| \leqslant C_{SL}\tilde{C}\left(1 + \|u\|^r_{L^{\frac{2d}{d+2}}(\Lambda)}\right)\|v\|_V.$$

Let $\lambda = \frac{1}{r} \in (0,1)$ and $\bar{q} = \frac{d+2}{d-2} = 5$, $\bar{q}' = \frac{d+2}{4} = \frac{5}{4}$. We infer

$$\begin{aligned}\|u\|^r_{L^q(\Lambda)} &\leqslant \|u\|^{\lambda r}_{L^{\lambda q\bar{q}}(\Lambda)}\|u\|^{(1-\lambda)r}_{L^{(1-\lambda)q\bar{q}'}} \\ &= \|u\|_{L^{\frac{2d\bar{q}}{d+2}}(\Lambda)}\|u\|^{r-1}_{L^{\frac{(r-1)q\bar{q}'}{r}}(\Lambda)} = \|u\|_{L^{\frac{2d}{d-2}}(\Lambda)}\|u\|^{r-1}_{L^{\frac{(r-1)d}{2}}(\Lambda)} \\ &= \|u\|_{L^6(\Lambda)}\|u\|^{\frac{4}{3}}_{L^2(\Lambda)} \leqslant C_{6,V}\|u\|_V\|u\|_H^{\frac{4}{3}},\end{aligned}$$

where $C_{6,V} > 0$ is the constant from the continuous embedding $H_0^{1,2}(\Lambda) \hookrightarrow L^6(\Lambda)$ as above and since $r = \frac{7}{3}$. Hence

$$\left| {}_{V^*}\langle f_0(u), v\rangle_V \right| \leqslant C_{SL}\tilde{C}^2 \left(1 + \|u\|_V \|u\|_H^{\frac{4}{3}}\right) \|v\|_V .$$

By Lemma A.3 we have

$$\left| {}_{V^*}\langle \Delta u, v\rangle_V \right| \leqslant \|u\|_V \|v\|_V .$$

Now by Lemma F.6 there exists a constant $C_4 = C_4(|\Lambda|) > 0$ such that

$$\begin{aligned}
\left| {}_{V^*}\langle \langle b, \nabla u\rangle_{\mathbb{R}^d}, v\rangle_V \right| &\leqslant \int_\Lambda |b| \, |\nabla u| \, |v| \, \mathrm{d}\xi \\
&\overset{F.6}{\leqslant} \left(\|u\|_V^2 + C_4 \|u\|_H^2\right)^{\frac{1}{2}} \left(\|v\|_V^2 + C_4 \|v\|_H^2\right)^{\frac{1}{2}} \\
&\overset{F.1\,(i)}{\leqslant} \left(\left(1 + C_4 C_{2,V}^2\right)^{\frac{1}{2}} \|u\|_V\right) \left(\left(1 + C_4 C_{2,V}^2\right)^{\frac{1}{2}} \|v\|_V\right) \\
&= \left(1 + C_4 C_{2,V}^2\right) \|u\|_V \|v\|_V ,
\end{aligned}$$

where $C_{2,V} > 0$ is the constant from the continuous embedding $V \hookrightarrow L^2(\Lambda) = H$. Altogether we have

$$\begin{aligned}
&\left| {}_{V^*}\langle A_{b,f}(u), v\rangle_V \right| \leqslant \left| {}_{V^*}\langle \Delta u, v\rangle_V \right| + \left| {}_{V^*}\langle \langle b, \nabla u\rangle_{\mathbb{R}^d}, v\rangle_V \right| + \left| {}_{V^*}\langle f_0(u), v\rangle_V \right| \\
&\leqslant \|u\|_V \|v\|_V + \left(1 + C_4 C_{2,V}^2\right) \|u\|_V \|v\|_V + C_{SL}\tilde{C}^2 \left(1 + \|u\|_V \|u\|_H^{\frac{4}{3}}\right) \|v\|_V \\
&= \|v\|_V \left(\left(2 + C_4 C_{2,V}^2\right) \|u\|_V + C_{SL}\tilde{C}^2 \left(1 + \|u\|_V \|u\|_H^{\frac{4}{3}}\right)\right) \\
&\leqslant \bar{C} \|v\|_V \left(\|u\|_V + \|u\|_V \|u\|_H^{\frac{4}{3}} + 1\right) \qquad (3.1.9)
\end{aligned}$$

with $\bar{C} = C_{SL}\tilde{C}^2 + C_4 C_{2,V}^2 + 2$ and we see that $A_{b,f_0}(u) \in V^*$.
Claim: (A1) holds.
Let $u, v, w \in V$ and $\lambda \in \mathbb{R}$ with $|\lambda| < 1$. We have to show that

$$\begin{aligned}
0 &= \lim_{\lambda \to 0} \left({}_{V^*}\langle A_{b,f_0}(u + \lambda v), w\rangle_V - {}_{V^*}\langle A_{b,f_0}(u), w\rangle_V \right) \\
&= \lim_{\lambda \to 0} \Bigg(\int_\Lambda \Big(\langle \Delta(u + \lambda v) + \langle b, \nabla(u + \lambda v)\rangle_{\mathbb{R}} + f_0(u + \lambda v), w\rangle \\
&\qquad - \langle \Delta u + \langle b, \nabla u\rangle_{\mathbb{R}} + f_0(u), w\rangle \Big) \, \mathrm{d}\xi \Bigg).
\end{aligned}$$

By the definition of A_b, we see that A_b is linear. Moreover, f_0 is assumed to be continuous in condition (SL2). So we have $\mathrm{d}\xi$-a.e. convergence to zero for the integrands. The claim then follows by Lebesgue's dominated convergence theorem. Since $|\lambda| < 1$ and $r = \frac{7}{3} \geqslant 1$ we have with Lemma B.3

$$\begin{aligned}
&\langle \Delta (u + \lambda v) + \langle b, \nabla (u + \lambda v)\rangle_{\mathbb{R}} + f_0 (u + \lambda v), w\rangle \\
&= \langle \Delta u, w\rangle + \lambda \langle \Delta v, w\rangle + \langle \langle b, \nabla u\rangle_{\mathbb{R}^d}, w\rangle + \lambda \langle \langle b, \nabla v\rangle_{\mathbb{R}^d}, w\rangle + \langle f_0 (u + \lambda v), w\rangle \\
&\leqslant \langle \Delta u, w\rangle + \langle \Delta v, w\rangle + |b| |\nabla u| |w| + |b| |\nabla v| |w| + |f_0 (u + \lambda v)| |w| \\
&\overset{(SL2)}{\leqslant} \langle \Delta u, w\rangle + \langle \Delta v, w\rangle + |b| |\nabla u| |w| + |b| |\nabla v| |w| + C_{SL} (1 + (|u| + |v|)^r) |w| \\
&\overset{B.3}{\leqslant} \langle \Delta u, w\rangle + \langle \Delta v, w\rangle + |b| |\nabla u| |w| + |b| |\nabla v| |w| + C_{SL} \left(1 + 2^{\frac{4}{3}} (|u|^r + |v|^r)\right) |w|
\end{aligned}$$

and this term dominates the integrands. It is only left to show that the term in the last row is integrable. Integration by parts and Hölder's inequality with $q = 2 = q'$ gives

$$\int_\Lambda \langle \Delta u, w\rangle \, \mathrm{d}\xi = -\int_\Lambda \langle \nabla u, \nabla v\rangle \, \mathrm{d}\xi \leqslant |\|u\|_V \|v\|_V| < \infty.$$

Lemma F.6 delivers for a constant $C_4 > 0$

$$\begin{aligned}
\int_\Lambda |b| |\nabla u| |w| \, \mathrm{d}\xi &\leqslant \left(\|w\|_V^2 + C_4 \|w\|_H^2\right)^{\frac{1}{2}} \left(\|u\|_V^2 + C_4 \|u\|_H^2\right)^{\frac{1}{2}} \\
&\leqslant \left(\left(1 + C_4 C_{2,V}^2\right) \|w\|_V^2\right)^{\frac{1}{2}} \left(\left(1 + C_4 C_{2,V}^2\right) \|u\|_V^2\right)^{\frac{1}{2}} \\
&= \left(1 + C_4 C_{2,V}^2\right) \|w\|_V \|u\|_V < \infty,
\end{aligned}$$

where $C_{2,V}$ is the constant from the continuous embedding $V \hookrightarrow L^2(\Lambda) = H$. Let $C_{1,V} > 0$ the constant from the embedding $V \hookrightarrow L^1(\Lambda)$, then

$$\int_\Lambda |w| \, \mathrm{d}\xi = \|w\|_{L^1(\Lambda)} \leqslant C_1 \|w\|_V < \infty$$

and for the last term we calculate

$$\begin{aligned}
\int_\Lambda |u|^r |w| \, \mathrm{d}\xi &\overset{Hölder}{\leqslant} \left(\int_\Lambda |u|^{2r} \, \mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_\Lambda |w|^2 \, \mathrm{d}\xi\right)^{\frac{1}{2}} = \|u\|_{L^{\frac{14}{3}}(\Lambda)}^{\frac{7}{3}} \|w\|_H \\
&\leqslant C_{2,V} C_{\frac{14}{3},V}^{\frac{7}{3}} \|u\|_V^{\frac{7}{3}} \|w\|_V < \infty,
\end{aligned}$$

where $C_{\frac{14}{3},V} > 0$ comes from the embedding $V \hookrightarrow L^{\frac{14}{3}}(\Lambda)$. This embedding exists by Theorem F.1 (i) since

$$0 - \frac{9}{14} \leqslant 1 - \frac{3}{2} \quad \Leftrightarrow \quad \frac{1}{2} \leqslant \frac{9}{14} = \frac{1}{2} + \frac{1}{7}.$$

Claim: (A2) holds.
Let $u, v \in V$. By (SL2) with $s = \frac{4}{3}$, Hölder's inequality with $q = 2$ and Lemma F.5 (ii) we get

$$\begin{aligned}
{}_{V^*}\langle f_0(u) - f_0(v), u - v\rangle_V &\leqslant C_{SL} \int_\Lambda (1 + |v|^s)(u-v)^2 \, \mathrm{d}x \\
&\leqslant C_{SL} \|u-v\|_H^2 + C_{SL} \int_\Lambda |v|^{\frac{4}{3}} |u-v|^2 \, \mathrm{d}x \\
&\overset{Hölder}{\leqslant} C_{SL} \|u-v\|_H^2 + C_{SL} \left(\int_\Lambda |v|^{\frac{4}{3}\cdot 2} \, \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Lambda |u-v|^{2\cdot 2} \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&= C_{SL} \|u-v\|_H^2 + C_{SL} \|v\|_{L^{\frac{8}{3}}(\Lambda)}^{\frac{4}{3}} \|u-v\|_{L^4(\Lambda)}^2 \\
&\overset{F.5\,(ii)}{\leqslant} C_{SL} \|u-v\|_H^2 + C_{SL} \|v\|_{L^{\frac{8}{3}}(\Lambda)}^{\frac{4}{3}} \cdot 2\sqrt{2} \|u-v\|_H^{\frac{1}{2}} \|u-v\|_V^{\frac{3}{2}} \\
&= C_{SL} \|u-v\|_H^2 + \left(\left(\frac{3}{2}\right)^{\frac{3}{4}} 2\sqrt{2} \cdot C_{SL} \|v\|_{L^{\frac{8}{3}}(\Lambda)}^{\frac{4}{3}} \|u-v\|_H^{\frac{1}{2}}\right) \left(\left(\frac{2}{3}\right)^{\frac{3}{4}} \|u-v\|_V^{\frac{3}{2}}\right)
\end{aligned}$$

Now by Young's inequality with $q = \frac{4}{3}$, $q' = 4$

$$\begin{aligned}
&\left(\left(\frac{3}{2}\right)^{\frac{3}{4}} 2\sqrt{2} \cdot C_{SL} \|v\|_{L^{\frac{8}{3}}(\Lambda)}^{\frac{4}{3}} \|u-v\|_H^{\frac{1}{2}}\right) \left(\left(\frac{2}{3}\right)^{\frac{3}{4}} \|u-v\|_V^{\frac{3}{2}}\right) \\
\leqslant &\frac{3}{4} \left(\left(\frac{2}{3}\right)^{\frac{3}{4}} \|u-v\|_V^{\frac{3}{2}}\right)^{\frac{4}{3}} + \frac{1}{4} \left(\left(\frac{3}{2}\right)^{\frac{3}{4}} 2\sqrt{2} \cdot C_{SL} \|v\|_{L^{\frac{8}{3}}(\Lambda)}^{\frac{4}{3}} \|u-v\|_H^{\frac{1}{2}}\right)^4 \\
= &\frac{1}{2} \|u-v\|_V^2 + 54 C_{SL}^4 \|v\|_{L^{\frac{8}{3}}(\Lambda)}^{\frac{16}{3}} \|u-v\|_H^2 .
\end{aligned}$$

Let us estimate $\|v\|_{L^{\frac{8}{3}}(\Lambda)}^{\frac{16}{3}}$. We repeat that $s = \frac{4}{3}$ and set $q = 6$, $q' = \frac{6}{5}$ and $\lambda = \frac{1}{2s} \in (0, 1)$. Then by Hölder's inequality

$$\begin{aligned}
\|v\|_{L^{\frac{8}{3}}(\Lambda)}^{\frac{16}{3}} &= \|v\|_{L^{2s}}^{4s} \leqslant \|v\|_{L^{2\lambda s q}}^{4\lambda s} \|v\|_{L^{s(1-\lambda)sq'}}^{4(1-\lambda)s} \\
&= \|v\|_{L^6}^2 \|v\|_{L^2}^{\frac{10}{3}} .
\end{aligned}$$

Since we have $d = 3$, we can use the well known Sobolev embedding $H_0^{1,p}(\Lambda) \hookrightarrow L^{\frac{dp}{d-p}}(\Lambda)$. So, for $p = 2$ there exists a constant $C_{6,V}$ such that

$$\|v\|_{L^6}^2 \leqslant C_{6,V}^2 \|v\|_V^2 .$$

Hence

$$\begin{aligned}
{}_{V^*}\langle f_0(u) - f_0(v), u - v\rangle_V &\leqslant \frac{1}{2} \|u-v\|_V^2 + 54 C_{SL}^4 C_{6,V}^2 \|v\|_V^2 \|v\|_H^{\frac{10}{3}} \|u-v\|_H^2 + C_{SL} \|u-v\|_H^2 \\
&\leqslant \frac{1}{2} \|u-v\|_V^2 + \left(C_{SL} + 54 C_{SL}^4 C_{6,V}^2 \|v\|_V^2 \|v\|_H^{\frac{10}{3}}\right) \|u-v\|_H^2 .
\end{aligned}$$

Lemma 3.1.2 (iii) gives

$$2\,{}_{V^*}\langle A_b(u) - A_b(v), u - v\rangle_V \leqslant -\|u - v\|_V^2 + C_3\|u - v\|_H^2.$$

By (SL3) we have

$$\|B(u) - B(v)\|_{L_2}^2 \leqslant C_{SL}\left(1 + \int_\Lambda |\nabla v|^2\, \mathrm{d}\xi\right)\int_\Lambda |u - v|^2\, \mathrm{d}\xi = C_{SL}\left(1 + \|v\|_V^2\right)\|u - v\|_H^2$$

and by (SL4)

$$\int_Z \|g(u,z) - g(v,z)\|_H^2\, m(\mathrm{d}z) \leqslant C_{SL}\int_\Lambda |u - v|^2\, \mathrm{d}\xi = C_{SL}\left(1 + \|v\|_V^2\right)\|u - v\|_H^2.$$

Therefore

$$\begin{aligned}
&2\,{}_{V^*}\langle A_{b,f_0}(u) - A_{b,f_0}(v), u - v\rangle_V + \|B(u) - B(v)\|_{L_2}^2 + \int_Z \|g(u,z) - g(v,z)\|_H^2\, m(\mathrm{d}z)\\
&\leqslant -\|u - v\|_V^2 + \|u - v\|_V^2 + C_3\|u - v\|_H^2 + 2C_{SL}\left(1 + \|v\|_V^2\right)\|u - v\|_H^2\\
&\quad + 2\left(C_{SL} + 54C_{SL}^4C_{6,V}^2\|v\|_V^2\|v\|_H^{\frac{10}{3}}\right)\|u - v\|_H^2\\
&= \left(C_3 + 2C_{SL}\left(1 + \|v\|_V^2\right) + 2\left(C_{SL} + 54C_{SL}^4C_{6,V}^2\|v\|_V^2\|v\|_H^{\frac{10}{3}}\right)\right)\|u - v\|_H^2\\
&\leqslant \varrho(v)\|u - v\|_H^2,
\end{aligned}$$

with $C_0 \geqslant \max\left\{C_3 + 4C_{SL}; 108 \cdot C_{SL}^4C_{6,V}^2\right\}$.
Claim: (A3) holds.
Let $u \in V$. Condition (SL2) with $f_0(0) = 0$ gives

$$2\,{}_{V^*}\langle f_0(u), u\rangle_V = 2\,{}_{V^*}\langle f_0(u) - f_0(0), u - 0\rangle_V \leqslant 2C_{SL}\int_\Lambda u^2\, \mathrm{d}\xi \leqslant 2C_{SL}\left(1 + \|u\|_H^2\right).$$

Then by Lemma 3.1.2 (iii) and Remark 3.1.1 (ii) we get

$$\begin{aligned}
&2\,{}_{V^*}\langle A_{b,f_0}(u), u\rangle_V + \|B(u)\|_{L_2}^2 + \|u\|_V^2 \leqslant C_3\|u\|_H^2 + (2C_{SL} + L)\left(1 + \|u\|_H^2\right)\\
&\leqslant (C_3 + 2C_{SL} + L)\left(1 + \|u\|_H^2\right) \leqslant F_t + K\|u\|_H^2
\end{aligned}$$

for all $t \in [0,T]$, and with $C_0 \geqslant C_3 + 2C_{SL} + L$.
Claim: (A4) holds.
Let $u \in V$. With $\alpha = 2$ we can calculate the operator norm of $A_{b,f_0}(u)$ by (3.1.9) and

Lemma B.3

$$\|A_{b,f_0}(u)\|_{V^*}^{\frac{\alpha}{\alpha-1}} = \left(\sup_{\substack{v\in V,\\ \|v\|_V=1}} \left| {}_{V^*}\langle A_{b,f_0}(u), v\rangle_V \right| \right)^2$$

$$\overset{(3.1.9)}{\leqslant} \bar{C}^2 \left(\|u\|_V + \|u\|_V \|u\|_H^{\frac{4}{3}} + 1 \right)^2 \overset{B.3}{\leqslant} 2\bar{C}^2 \left(1 + \left(\|u\|_V + \|u\|_V \|u\|_H^{\frac{4}{3}} \right)^2 \right)$$

$$\overset{B.3}{\leqslant} 4\bar{C}^2 \left(1 + \|u\|_V^2 + \|u\|_V^2 \|u\|_H^{\frac{8}{3}} \right) \leqslant 4\bar{C}^2 \left(1 + \|u\|_V^2 + \|u\|_V^2 \|u\|_H^{\frac{8}{3}} + \|u\|_H^{\frac{8}{3}} \right)$$

and with Lemma A.2 and B.3 we get

$$\|A_{b,f_0}(u)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leqslant 4\bar{C}^2 \left(1 + \|u\|_V^2 \right) \left(1 + \|u\|_H^{\frac{8}{3}} \right)$$

$$\overset{A.2}{\leqslant} 4\bar{C}^2 \left(1 + \|u\|_V^2 \right) \left(1 + \|u\|_H^{\frac{8}{3}} \right)^{\frac{5}{4}} \overset{B.3}{\leqslant} 2^{\frac{9}{4}} \bar{C}^2 \left(1 + \|u\|_V^2 \right) \left(1 + \|u\|_H^{\frac{10}{3}} \right)$$

$$\leqslant \left(F_t + K \|v\|_V^\alpha \right) \left(1 + \|v\|_H^\beta \right)$$

for all $t \in [0,T]$, where $F_t := C_0 =: K$, $\beta = \frac{10}{3}$ and $C_0 \geqslant 2^{\frac{9}{4}} \bar{C}^2$.
Claim: (B1) holds.
See the proof of (B1) in Example 3.1.3.
Claim: (B2) holds.
This follows from (3.1.7), since $\zeta = \frac{8}{3} = \frac{1}{2} \cdot \frac{16}{3} = \frac{\beta+2}{2}$.
Claim: (B3) holds.
By the definition of ϱ, α, β and C we have for all $v \in V$

$$\varrho(v) = C_0 \left(1 + \|v\|_V^2 \right) \left(1 + \|v\|_H^{\frac{10}{3}} \right) = C \left(1 + \|v\|_V^\alpha \right) \left(1 + \|v\|_H^\beta \right).$$

Claim: (3.1.1) with $A = A_{b,f_0}$ has a solution.
Since all conditions are fulfilled, we can apply Theorem 2.2.1 to get a solution $X = (X_t)_{t\in[0,T]}$ to (3.1.1) and by 2.2.1 (i) we have

$$\sup_{t\in[0,T]} \mathbb{E} \left[\|X(t)\|_H^{\frac{16}{3}} \right] < \infty.$$

Claim: If $\eta = 0$, then the solution is unique.
Suppose $\eta = \gamma = 0$, then Theorem 2.2.1 (ii) is applicable and we obtain uniqueness and

$$\mathbb{E} \left[\sup_{t\in[0,T]} \|X(t)\|_H^{\frac{16}{3}} \right] < \infty.$$

□

3.1.6 Remark. *It is not necessary to claim $\eta = 0$ to get the uniqueness result in Examples 3.1.3, 3.1.4 and 3.1.5. A sufficient condition is $0 \leqslant \eta < \Gamma$, where Γ is the constant from (2.2.1) (cf. page 17), i.e.*

$$0 \leqslant \eta < \begin{cases} \left[2\left(9C_{BDG}^2 + 53\right)\right]^{-1}, & \text{in Example 3.1.3,} \\ \left[16C_{BDG}^2 + 46\sqrt[3]{2} + \frac{26}{3}\right]^{-1}, & \text{in Example 3.1.5,} \end{cases}$$

and, in Example 3.1.4,

$$0 \leqslant \eta < \begin{cases} \left[12C_{BDG}^2 + 26\right]^{-1}, & \text{if } r \leqslant 2, \\ 2r\left[r^2\left(12C_{BDG}^2 + 8\right) + r\left(4^{r+1} - 4\right) - 3 \cdot 4^r\right]^{-1}, & \text{else.} \end{cases}$$

3.2. Quasi-linear stochastic equations: p-Laplacian

In this section let $d \in \mathbb{N}$ with $d \geqslant 3$. Again, let $\Lambda \subset \mathbb{R}^d$ be an open, bounded domain. Let $2 \leqslant p < \infty$. We have the Gelfand triple

$$V := H_0^{1,p}(\Lambda) \subset H := L^2(\Lambda) \subset H_0^{-1,p}(\Lambda) = V^*.$$

Consider the following equation:

$$\begin{aligned} \mathrm{d}X(t) = {} & \left(\sum_{i=1}^d D_i\left(|D_i X(t)|^{p-2} D_i X(t)\right) + f_0(X(t))\right) \mathrm{d}t + B(X(t))\,\mathrm{d}W(t) \\ & + \int_Z f(X(t-), z)\,\bar{\mu}(\mathrm{d}t, \mathrm{d}z), \\ X(0) = {} & X_0. \end{aligned} \tag{3.2.1}$$

Suppose that there exist $C_{QL}, r, s \geqslant 0$ such that the following holds:

(QL1) $f_0 \colon \mathbb{R} \to \mathbb{R}$ is continuous, $f_0(0) = 0$ and f_0 satisfies for all $x, y \in \mathbb{R}$

$$|f_0(x)| \leqslant C_{QL}(1 + |x|^r),$$
$$(f_0(x) - f_0(y))(x - y) \leqslant C_{QL}(1 + |y|^s)|x - y|^h.$$

(QL2) $B \colon H_0^{1,p}(\Lambda; \mathbb{R}) \to L_2\left(U; L^2(\Lambda; \mathbb{R})\right)$ satisfies for all $v_1, v_2 \in H_0^{1,p}(\Lambda; \mathbb{R})$

$$\|B(v_1) - B(v_2)\|_{L_2}^2 \leqslant C_{QL}\left(1 + \int_\Lambda |\nabla v_2|^p\,\mathrm{d}x\right)\int_\Lambda |v_1 - v_2|^2\,\mathrm{d}x.$$

(QL3) $f \colon \mathbb{R} \times Z \to \mathbb{R}$ such that for all $v, v_1, v_2 \in H_0^{1,p}(\Lambda; \mathbb{R})$ we have

$$\int_Z \int_\Lambda |f(v_1, z) - f(v_2, z)|^2\,\mathrm{d}x\,m(\mathrm{d}z) \leqslant C_{QL}\left(1 + \int_\Lambda |\nabla v_2|^p\,\mathrm{d}x\right)\int_\Lambda |v_1 - v_2|^2\,\mathrm{d}x,$$

$$\begin{aligned} \int_Z \left(\int_\Lambda |f(v, z)|^2\,\mathrm{d}x\right)^\zeta m(\mathrm{d}z) \leqslant {} & C_{QL}\left(1 + \left(\int_\Lambda |v|^2\,\mathrm{d}x\right)^\zeta\right) \\ & + \eta\left(\int_\Lambda |v|^2\,\mathrm{d}\xi\right)^{\zeta-1}\left(\int_\Lambda |\nabla v|^p\,\mathrm{d}\xi\right), \end{aligned}$$

$$\text{where } \zeta \geqslant 1 \text{ and } 0 \leqslant \eta < \begin{cases} 2^{2-p}\left(2\zeta + 4^{\zeta}\right)^{-1}, & \text{if } d < p, \\ 2^{1-p}\left(2\zeta + 4^{\zeta}\right)^{-1}, & \text{else.} \end{cases}$$

We define the operators

$$A_p(u) := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right),$$
$$A_{p,f_0}(u) := A_p(u) + f_0(u).$$

3.2.1 Lemma. *There exists a constant $C_1 = C_1(p) > 0$ such that for all $u, v \in V$ we have*

$$_{V^*}\langle A_p(u) - A_p(v), u - v\rangle_V \leqslant -C_1 \|u - v\|_V^p.$$

Proof. Let $u, v \in V$. By Lemma C.4 we have

$$\begin{aligned} &_{V^*}\langle A_p(u) - A_p(v), u - v\rangle_V \\ &= -\int_\Lambda \left\langle |\nabla u(x)|^{p-2} \nabla u(x) - |\nabla v(x)|^{p-2} \nabla v(x), \nabla u(x) - \nabla v(x) \right\rangle_{\mathbb{R}^d} \mathrm{d}x \\ &\overset{C.4}{\leqslant} -2^{-(p-2)} \int_\Lambda |\nabla u(x) - \nabla v(x)|^p \,\mathrm{d}x = -C_1 \|u - v\|_V^p, \end{aligned}$$

where $C_1 = C_1(p) = 2^{-(p-2)}$.

□

3.2.2 Remark. *As in Remark 3.1.1 (ii) and (iii) there exist constants $L_B, L_f > 0$ such that for all $v \in V$ we have*

$$\|B(v)\|_{L_2}^2 \leqslant L_B\left(1 + \|v\|_H^2\right),$$
$$\int_Z \|f(v,z)\|_H^2 \, m(\mathrm{d}z) \leqslant L_f\left(1 + \|v\|_H^2\right).$$

3.2.1. Examples

3.2.3 Example ($d < p$)**.** Let $d < p$. Suppose conditions (QL1) to (QL3) hold for $1 \leqslant s \leqslant p$, $r := p - 1$ and $h := 2$. Then for any initial value $X_0 \in L^{\bar{p}}(\Omega, \mathcal{F}_0, P; H)$, where $\bar{p} \geqslant 2\zeta$, equation (3.2.1) has a solution $X = (X_t)_{t\in[0,T]}$ and this solutions satisfies

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X(t)\|_H^{2\zeta}\right] < \infty.$$

If $\eta = 0$, then this solution is unique and we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|_H^{2\zeta}\right] < \infty.$$

Proof. Similarily to the proof of Example 3.1.3, this proof is divided into claims to verify conditions (A1)–(A4) and (B1)–(B3) and to show that $A_{p,f_0}\colon V \to V^*$. Finally, we will use these claims to show that there exists a solution and that this solution is unique. We will verify Theorem 2.2.1 for

$$\begin{aligned} \alpha &= p, & \beta &= 2(\zeta - 1), & \gamma &= \eta, \\ \theta &= 2C_1, & K &:= C_0, & C &:= C_0 \end{aligned}$$

and $F :\equiv C_0$, $\varrho(v) := C_0\left(1 + \|v\|_V^p\right)$ for $v \in V$, where $C_1 = C_1(p) = 2^{2-p} > 0$ is the constant from Lemma 3.2.1 and $C_0 > 0$ is big enough, which we will see in the following proof. Let us note that $\beta = 2(\zeta - 1) \geqslant 0$.
Claim: $\gamma < \theta^{\frac{\beta+2}{2}} \cdot \left[(\beta+2)(\beta+1) + 2^{\beta+1}(2\beta+1)\right]^{-1}$.
Since

$$\gamma = \eta < 2^{2-p}\left(2\zeta + 4^{\zeta}\right)^{-1} = \frac{\theta}{2}\left[(\beta+2) + 2^{\beta+2}\right]^{-1},$$

the assertion follows from Remark 2.2.2.
Claim: $A_{p,f_0}\colon V \to V^*$.
Let $u, v \in V$. By condition (QL1) with $r = p - 1$

$$\begin{aligned} &\left|{}_{V^*}\langle f_0(u), v\rangle_V\right| \leqslant C_{QL}\int_\Lambda (1 + |u|^r)\,|v|\ \mathrm{d}\xi \\ \leqslant& C_{QL}\int_\Lambda \left(1 + \left(\sup_\Lambda |u|\right)^r\right)|v|\ \mathrm{d}\xi = C_{QL}\left(1 + \|u\|_{L^\infty(\Lambda)}^r\right)\|v\|_{L^1(\Lambda)} \\ \leqslant& C_{QL}\left(1 + C_{\infty,V}^{p-1}\|u\|_V^{p-1}\right)\|v\|_{L^1(\Lambda)}, \end{aligned}$$

where $C_{\infty,V} = C_{\infty,V}(p, d, |\Lambda|) > 0$ from Proposition F.3. With $C_{1,V} = C_{1,V}(p, d, |\Lambda|) > 0$ we also have

$$\|v\|_{L^1(\Lambda)} \leqslant C_{1,V}\|v\|_V.$$

Furthermore we have by the Cauchy-Schwarz inequality and Hölder's inequality

$$\begin{aligned} &\left|{}_{V^*}\langle A_p(u), v\rangle_V\right| \leqslant \left|\int_\Lambda \left\langle |\nabla u|^{p-2}\nabla u, \nabla v\right\rangle_{\mathbb{R}^d}\ \mathrm{d}\xi\right| \\ \leqslant& \int_\Lambda \left||\nabla u|^{p-2}\nabla u\right||\nabla v|\ \mathrm{d}\xi = \int_\Lambda |\nabla u|^{p-1}|\nabla v|\ \mathrm{d}\xi \\ \overset{\text{Hölder}}{\leqslant}& \left(\int_\Lambda |\nabla u|^{(p-1)\frac{p}{p-1}}\ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\left(\int_\Lambda |\nabla v|^p\ \mathrm{d}\xi\right)^{\frac{1}{p}} = \|u\|_V^{p-1}\|v\|_V. \end{aligned} \tag{3.2.2}$$

Hence

$$\left|{}_{V^*}\langle A_{p,f_0}(u), v\rangle_V\right| \leqslant \left(C_{QL}C_{1,V} + \left(1 + C_{QL}C_{1,V}C_{\infty,V}^{p-1}\right)\|u\|_V^{p-1}\right)\|v\|_V, \tag{3.2.3}$$

and we conclude $A_{p,f_0}(u) \in V^*$.

Claim: (A1) holds.
Let $u,v,w\in V$ and $\lambda\in\mathbb{R}$ with $|\lambda|\leqslant 1$. We have to show

$$0=\lim_{\lambda\to 0}\left({}_{V^*}\langle A_{p,f_0}(u+\lambda v),w\rangle_V-{}_{V^*}\langle A_{p,f_0}(u),w\rangle_V\right)$$
$$=\lim_{\lambda\to 0}\left(\int_\Lambda\left(\left\langle|\nabla(u+\lambda v)|^{p-2}\nabla(u+\lambda v)+f_0(u+\lambda v),\nabla w\right\rangle-\left\langle|\nabla u|^{p-2}\nabla u+f_0(u),\nabla w\right\rangle\right)\mathrm{d}\xi\right).$$

The integrands converge to 0 dξ-a.s. since f_0 is continuous. (A1) follows by Lebesgue's monotone convergence theorem, because by the Cauchy-Schwarz inequality, condition (QL1) with $r=p-1$, Lemma B.3 and $|\lambda|\leqslant 1$ we have

$$\begin{aligned}
&\left|\left\langle|\nabla(u+\lambda v)|^{p-2}\nabla(u+\lambda v)+f_0(u+\lambda v),\nabla w\right\rangle\right|\\
\leqslant&\left(|\nabla(u+\lambda v)|^{p-1}+|f_0(u+\lambda v)|\right)|\nabla w|\\
\leqslant&\left(|\nabla u+\lambda\nabla v|^{p-1}+C_{QL}\left(1+|u+\lambda v|^{p-1}\right)\right)|\nabla w|\\
\overset{B.3}{\leqslant}&2^{p-2}\left(|\nabla u|^{p-1}+|\lambda\nabla v|^{p-1}+C_{QL}\left(|u|^{p-1}+|\lambda v|^{p-1}\right)+C_{QL}2^{2-p}\right)|\nabla w|\\
\leqslant&2^{p-2}\left(|\nabla u|^{p-1}+|\nabla v|^{p-1}+C_{QL}\left(|u|^{p-1}+|v|^{p-1}\right)+C_{QL}2^{2-p}\right)|\nabla w|
\end{aligned}$$

and this term dominates the integrands. We still have to show that this term is integrable.

$$\int_\Lambda|\nabla u|^{p-1}|\nabla w|\,\mathrm{d}\xi\overset{\text{Hölder}}{\leqslant}\|u\|_V^{p-1}\|w\|_V<\infty$$

and

$$\begin{aligned}
\int_\Lambda|u|^{p-1}|\nabla w|\,\mathrm{d}\xi&\overset{\text{Hölder}}{\leqslant}\left(\int_\Lambda|u|^p\,\mathrm{d}\xi\right)^{\frac{p-1}{p}}\|w\|_V=\|u\|_{L^p(\Lambda)}^{p-1}\|w\|_V\\
&\leqslant C_{p,V}^{p-1}\|u\|_V^{p-1}\|w\|_V<\infty,
\end{aligned}$$

where $C_{p,V}=C_{p,V}(p,d,|\Lambda|)>0$ is the constant from Poincaré's inequality F.4.
Claim: (A2) holds.
Let $u,v\in V$. Condition (QL1) with $h=2$ gives us

$$\begin{aligned}
{}_{V^*}\langle f_0(u)-f_0(v),u-v\rangle_V&=\int_\Lambda(f_0(u)-f_0(v))(u-v)\,\mathrm{d}\xi\\
&\overset{(QL1)}{\leqslant}C_{QL}\int_\Lambda(1+|v|^s)|u-v|^2\,\mathrm{d}\xi\leqslant C_{QL}\left(1+\|v\|_{L^\infty(\Lambda)}^s\right)\|u-v\|_H^2.
\end{aligned}$$

Now by Proposition F.3 we have $\|v\|_{L^\infty(\Lambda)}\leqslant C_{\infty,V}\|v\|_V$, where $C_{\infty,V}=C_{\infty,V}(d,p,|\Lambda|)>0$. Young's inequality leads to

$$\begin{aligned}
C_{QL}\left(1+\|v\|_{L^\infty(\Lambda)}^s\right)\|u-v\|_H^2&\leqslant C_{QL}\left(1+C_{\infty,V}\|v\|_V^s\right)\|u-v\|_H^2\\
&\overset{Young}{\leqslant}C_{QL}\left(1+\frac{p-s}{p}C_{\infty,V}^{\frac{p}{p-s}}+\frac{s}{p}\|v\|_V^p\right)\|u-v\|_H^2\\
&\leqslant C_{QL}\left(1+\frac{p-s}{p}C_{\infty,V}^{\frac{p}{p-s}}\right)\left(1+\frac{s}{p}\right)\left(1+\|v\|_V^p\right)\|u-v\|_H^2.
\end{aligned}$$

By Lemma 3.2.1 and conditions (QL2) and (QL3) we get

$$
\begin{aligned}
&2\,{}_{V^*}\langle A_{p,f_0}(u) - A_{p,f_0}(v), u - v\rangle_V + \|B(u) - B(v)\|_{L_2}^2 + \int_Z \|f(u,z) - f(v,z)\|_H^2 \, m(\mathrm{d}z) \\
\leqslant & \underbrace{-2C_1 \|u - v\|_V^p}_{\leqslant 0} + 3C_{QL}\left(1 + \int_\Lambda |\nabla v|^p \, \mathrm{d}x\right) \int_\Lambda |u - v|^2 \, \mathrm{d}x \\
&+ 2C_{QL}\left(1 + \frac{p-s}{p} C_{\infty,V}^{\frac{p}{p-s}}\right)\left(1 + \frac{s}{p}\right)\left(1 + \|v\|_V^p\right)\|u - v\|_H^2 \\
\leqslant & 3C_{QL}\left(1 + \frac{p-s}{p} C_{\infty,V}^{\frac{p}{p-s}}\right)\left(1 + \frac{s}{p}\right)\left(1 + \|v\|_V^p\right)\|u - v\|_H^2 \\
\leqslant & \left(F_t + \varrho(v)\right)\|u - v\|_H^2,
\end{aligned}
$$

for all $t \in [0,T]$, since $F \equiv C_0$ and $C_0 \geqslant 3C_{QL}\left(1 + \frac{p-s}{p} C_{\infty,V}^{\frac{p}{p-s}}\right)\left(1 + \frac{s}{p}\right)$.

Claim: (A3) holds.

Let $v \in V$. Then by condition (QL1), since $f_0(0) = 0$,

$$
\begin{aligned}
{}_{V^*}\langle f_0(v), v\rangle_V &= {}_{V^*}\langle f_0(v) - f_0(0), v - 0\rangle_V = \int_\Lambda \left(f_0(v) - f_0(0)\right)(v - 0) \, \mathrm{d}\xi \\
&\overset{\text{(QL1)}}{\leqslant} C_{QL} \int_\Lambda |v|^2 \, \mathrm{d}\xi = C_{QL}\|v\|_H^2.
\end{aligned}
$$

By Remark 3.2.2 and Lemma 3.2.1 we conclude

$$
\begin{aligned}
&2\,{}_{V^*}\langle A_{p,f_0}(v), v\rangle_V + \|B(v)\|_{L_2}^2 + 2C_1\|v\|_V^p \leqslant L_B\left(1 + \|v\|_H^2\right) + 2C_{QL}\|v\|_H^2 \\
\leqslant & L_B + (L_B + 2C_{QL})\|v\|_H^2 \leqslant F_t + K\|v\|_H^2,
\end{aligned}
$$

for all $t \in [0,T]$, since $F_t \equiv C_0$, $K = C_0$ and $C_0 \geqslant L_B + 2C_{QL}$.

Claim: (A4) holds.

Let $u \in V$. By the definition of the norm of the operator $A_{p,f_0}(u) : V \to \mathbb{R}$, with $\alpha = p$ and with (3.2.3)

$$
\begin{aligned}
\|A_{p,f_0}(u)\|_{V^*}^{\frac{\alpha}{\alpha-1}} &= \left[\sup_{\substack{v \in V, \\ \|v\|_V = 1}} \left|{}_{V^*}\langle A_{p,f_0}(u), v\rangle_V\right|\right]^{\frac{p}{p-1}} \\
&\leqslant \left[C_{QL}C_{1,V} + \left(1 + C_{QL}C_{1,V}C_{\infty,V}^{p-1}\right)\|u\|_V^{p-1}\right]^{\frac{p}{p-1}} \\
&\leqslant \tilde{C}\left(1 + \|u\|_V^{p-1}\right)^{\frac{p}{p-1}},
\end{aligned}
$$

where $\tilde{C} = \left[\max\left\{C_{QL}C_{1,V}; 1 + C_{QL}C_{1,V}C_{\infty,V}^{p-1}\right\}\right]^{\frac{p}{p-1}}$. Since $\frac{p}{p-1} \geqslant 1$ we have by Lemma

B.3

$$\begin{aligned}\|A_{p,f_0}(u)\|_{V^*}^{\frac{\alpha}{\alpha-1}} &\leqslant \tilde{C}\left(1+\|u\|_V^{p-1}\right)^{\frac{p}{p-1}}\\ &\overset{B.3}{\leqslant} 2^{-(p-1)}\tilde{C}\left(1+\|u\|_V^p\right)\\ &\leqslant \left(F_t + K\|u\|_V^\alpha\right)\underbrace{\left(1+\|u\|_H^\beta\right)}_{\geqslant 1}\end{aligned}$$

for all $t \in [0,T]$ with $K := C_0 \equiv: F_t$ and $C_0 \geqslant 2^{-(p-1)}\tilde{C}$.
Claim: (B1) holds.
Let $u \in V$. Then by Remark 3.2.2

$$\|B(v)\|_{L_2}^2 + \int_Z \|f(v,z)\|_H^2\, m(\mathrm{d}z) \leqslant 2\left(L_B + L_f\right)\left(1+\|v\|_H^2\right) \leqslant C\left(1 + F_t + \|v\|_H^2\right),$$

for all $t \in [0,T]$, since F is non-negative and $C := C_0 \geqslant 2\left(L_B + L_f\right)$.
Claim: (B2) holds.
Let $u \in V$. Since $\beta = 2(\zeta - 1)$ and $\alpha = p$, we have together with condition (QL3)

$$\begin{aligned}&\int_Z \|f(v,z)\|_H^{\beta+2}\, m(\mathrm{d}z) = \int_Z\left(\int_\Lambda |f(v,z)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}(\beta+2)} m(\mathrm{d}z)\\ &= \int_Z\left(\int_\Lambda |f(v,z)|^2\,\mathrm{d}\xi\right)^{\zeta} m(\mathrm{d}z)\\ &\overset{(QL3)}{\leqslant} C_{QL}\left(1+\left(\int_\Lambda |v|^2\,\mathrm{d}\xi\right)^{\zeta}\right) + \eta\left(\int_\Lambda |v|^2\,\mathrm{d}\xi\right)^{\zeta-1}\left(\int_\Lambda |\nabla v|^p\,\mathrm{d}\xi\right)\\ &= C_{QL}\left(1+\|v\|_H^{2\zeta}\right) + \eta\|v\|_H^{2(\zeta-1)}\|v\|_V^p = C_{QL}\left(1+\|v\|_H^{\beta+2}\right) + \eta\|v\|_H^\beta\|v\|_V^\alpha\\ &\leqslant C\left(1 + F_t^{\frac{\beta+2}{2}} + \|v\|_H^{\beta+2}\right) + \eta\|v\|_H^\beta\|v\|_V^\alpha\end{aligned}$$

for all $t \in [0,T]$ since $F :\equiv C_0$ is non-negative and $C := C_0 \geqslant C_{QL}$.
Claim: (B3) holds.
This is clear by the definition of ϱ and α: For $v \in V$ we have

$$\varrho(v) = C_0\left(1+\|v\|_V^p\right) \leqslant C\left(1+\|v\|_V^p\right)\underbrace{\left(1+\|v\|_H^\beta\right)}_{\geqslant 1}.$$

Claim: (3.2.1) has a solution.
Theorem 2.2.1 gives us a solution $X = (X_t)_{t\in[0,T]}$. With 2.2.1 (i) we have

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X_t\|_H^{2\zeta}\right] < \infty.$$

Claim: If $\eta = 0$, then the solution is unique.
Suppose $\gamma = \eta = 0$. Then by 2.2.1 (ii) our solution is unique and we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^{2\zeta}\right] < \infty.$$

□

3.2.4 Example ($d > p$)**.** Let $d > p$. Suppose conditions (QL1) to (QL3) hold for $1 \leqslant s \leqslant \min\left\{\frac{p(p-h)}{p-2}; \frac{(h-2)p^2}{(p-2)(d-p)}\right\}$, $r := p-1$ and $2 < h < p$. Then for any initial value $X_0 \in L^{\bar{p}}(\Omega, \mathcal{F}_0, P; H)$, where $\bar{p} \geqslant 2\zeta$, equation (3.2.1) has a solution $X = (X_t)_{t\in[0,T]}$ and this solutions satisfies

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X(t)\|_H^{2\zeta}\right] < \infty.$$

If $\eta = 0$, then this solution is unique and we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|_H^{2\zeta}\right] < \infty.$$

Proof. The structure of this proof is identical to the proof of Example 3.2.3. The conditions of Theorem 2.2.1 will be verified for

$$\begin{aligned} &\alpha = p, \qquad \beta = 2(\zeta - 1), \quad \gamma = \eta, \\ &\theta = C_1, \quad K := C_0, \qquad\quad C := C_0 \end{aligned}$$

and $F :\equiv C_0$, $\varrho(v) := C_0\left(1 + \|v\|_V^p\right)$ for $v \in V$, where $C_1 = C_1(p) = 2^{2-p} > 0$ is the constant from Lemma 3.2.1. $C_0 > 0$ is a constant big enough. We will see in the following proof how big C_0 has to be.
Claim: $\gamma < \theta^{\frac{\beta+2}{2}} \cdot \left[(\beta+2)(\beta+1) + 2^{\beta+1}(2\beta+1)\right]^{-1}$.
Since

$$\gamma = \eta < 2^{1-p}\left(2\zeta + 4^{\zeta}\right)^{-1} = \frac{\theta}{2}\left[(\beta+2) + 2^{\beta+2}\right]^{-1},$$

the assertion follows from Remark 2.2.2.
Claim: $A_{p,f_0} \colon V \to V^*$.
Let $u, v \in V$. Condition (QL1) with $r = p-1$ gives us together with Hölder's inequality for $q = \frac{p}{p-1}$, $q' = p$

$$\begin{aligned} \left|{}_{V^*}\langle f_0(u), v\rangle_V\right| &\leqslant C_{QL}\int_\Lambda (1 + |u|^r)\,|v|\;\mathrm{d}\xi \\ &\leqslant C_{QL}\left(\|v\|_{L^1(\Lambda)} + \int_\Lambda |u|^{p-1}\,|v|\;\mathrm{d}\xi\right) \\ &\overset{Hölder}{\leqslant} C_{QL}\left(\|v\|_{L^1(\Lambda)} + \|u\|_{L^p(\Lambda)}^{p-1}\,\|v\|_{L^p(\Lambda)}\right). \end{aligned}$$

Let $C_{p,V} > 0$ the constant from Poincaré's inequality F.4 for $V \hookrightarrow L^p(\Lambda)$ and let $C_{1,V} > 0$ the constant from Theorem F.1 for $V \hookrightarrow L^1(\Lambda)$ (since $0 \leqslant 1 + d\left(1 - \frac{1}{p}\right)$ holds true). Then we can estimate

$$\begin{aligned} \left|{}_{V^*}\langle f_0(u), v\rangle_V\right| &\leqslant C_{QL}\left(\|v\|_{L^1(\Lambda)} + \|u\|_{L^p(\Lambda)}^{p-1}\|v\|_{L^p(\Lambda)}\right) \\ &\leqslant C_{QL}\left(C_{p,V}^2 + C_{1,V}\right)\left(1 + \|u\|_V^{p-1}\right)\|v\|_V . \end{aligned}$$

Together with (3.2.2) we obtain

$$\begin{aligned} \left|{}_{V^*}\langle A_{p,f_0}(u), v\rangle_V\right| &\leqslant \|u\|_V^{p-1}\|v\|_V + C_{QL}\left(C_{p,V}^2 + C_{1,V}\right)\left(1 + \|u\|_V^{p-1}\right)\|v\|_V , \\ &\leqslant \left(C_{QL}\left(C_{p,V}^2 + C_{1,V}\right) + 1\right)\left(1 + \|u\|_V^{p-1}\right)\|v\|_V \end{aligned} \tag{3.2.4}$$

and we conclude $A_{p,f_0}(u) \in V^*$.
Claim: (A2) holds.
By Theorem F.1 (i) we have $V \hookrightarrow L^a(\Lambda)$ continuously for all $1 \leqslant a \leqslant \frac{dp}{d-p}$, because $d > p$. We define

$$\lambda := \frac{p(h-2)}{h(p-2)}, \quad p_0 := \frac{dp}{d-p}.$$

Then $\lambda \in (0,1)$ and $p_0 > 2$, because $2 < h < p$. Let p_λ such that

$$\frac{1}{p_\lambda} = \frac{1-\lambda}{2} + \frac{\lambda}{p_0}.$$

Especially we have $p_\lambda \in (2, p_0)$. To see that $h < p_\lambda$ we refer to Lemma A.1. Now let $u, v \in V$. Condition (QL1) gives us with Hölder's inequality for $q = \frac{p_\lambda}{p_\lambda - h}$, $q' = \frac{p_\lambda}{h}$

$$\begin{aligned} {}_{V^*}\langle f_0(u) - f_0(v), u - v\rangle_V &= \int_\Lambda \left(f_0(u) - f_0(v)\right)(u - v)\,\mathrm{d}\xi \\ \overset{(QL1)}{\leqslant} C_{QL}\int_\Lambda \left(1 + |v|^s\right)|u - v|^h\,\mathrm{d}\xi &= C_{QL}\left[\int_\Lambda 1 \cdot |u - v|^h\,\mathrm{d}\xi + \int_\Lambda |v|^s|u - v|^h\,\mathrm{d}\xi\right] \\ \overset{\text{Hölder}}{\leqslant} C_{QL}\left[|\Lambda|^{\frac{p_\lambda - h}{p_\lambda}}\|u - v\|_{L^{p_\lambda}(\Lambda)}^h + \|v\|_{L^{\frac{sp_\lambda}{p_\lambda - h}}(\Lambda)}^s\|u - v\|_{L^{p_\lambda}(\Lambda)}^h\right] \\ \leqslant C_{QL,\Lambda}\|u - v\|_{L^{p_\lambda}(\Lambda)}^h\left(1 + \|v\|_{L^{\frac{sp_\lambda}{p_\lambda - h}}(\Lambda)}^s\right), \end{aligned}$$

where $C_{QL,\Lambda} = C_{QL}\left(1 + |\Lambda|^{\frac{p_\lambda - h}{p_\lambda}}\right)$. By our choice of λ and p_λ we can apply Lemma F.7 and get

$$\begin{aligned} \|u - v\|_{L^{p_\lambda}(\Lambda)}^h &\leqslant \|u - v\|_{L^2(\Lambda)}^{h(1-\lambda)}\|u - v\|_{L^{p_0}(\Lambda)}^{\lambda h} \\ &\leqslant C_{p_0,V}^{\lambda h}\|u - v\|_{L^2(\Lambda)}^{h(1-\lambda)}\|u - v\|_V^{\lambda h}, \end{aligned}$$

where $C_{p_0,V} > 0$ is the constant from the continuous embedding $V \hookrightarrow L^{p_0}$. Let $C_1 > 0$ the constant from Lemma 3.2.1 and set

$$\bar{C}_1 := \frac{p}{\lambda h} C_1^{\frac{\lambda h}{p}} \quad \left(= \frac{p-2}{h-2} C_1^{\frac{h-2}{p-2}} \right).$$

Then by multiplying with $1 = \bar{C}_1 \cdot \bar{C}_1^{-1}$

$$\begin{aligned}
&{}_{V^*}\langle f_0(u) - f_0(v), u - v\rangle_V \\
&\leqslant \left[\bar{C}_1 \|u-v\|_V^{\lambda h}\right] \cdot \left[\bar{C}_1^{-1} C_{QL,\Lambda} C_{p_0,V}^{\lambda h} \|u-v\|_{L^2(\Lambda)}^{h(1-\lambda)} \left(1 + \|v\|^s_{L^{\frac{s p_\lambda}{p_\lambda - h}}(\Lambda)}\right)\right] \\
&\overset{Young}{\leqslant} C_1 \|u-v\|_V^p + \tilde{C} \left[\|u-v\|_{L^2(\Lambda)}^{h(1-\lambda)} \left(1 + \|v\|^s_{L^{\frac{s p_\lambda}{p_\lambda - h}}(\Lambda)}\right)\right]^{\frac{p-2}{p-h}}
\end{aligned}$$

where we used Young's inequality with $q = \frac{p}{\lambda h} = \frac{p-2}{h-2}$, $q' = \frac{p}{p-\lambda h} = \frac{p-2}{p-h}$ and

$$\tilde{C} = \frac{p-h}{p-2} \left(\bar{C}_1^{-1} C_{QL,\Lambda} C_{p_0,V}^{\lambda h}\right)^{\frac{p-2}{p-h}}.$$

We calculate

$$\left(\|u-v\|_{L^2(\Lambda)}^{h(1-\lambda)}\right)^{\frac{p-2}{p-h}} = \|u-v\|^2_{L^2(\Lambda)}.$$

Since $2 < h < p$ we have $\frac{p-2}{p-h} \geqslant 1$ and Lemma B.3 gives

$$\left(1 + \|v\|^s_{L^{\frac{s p_\lambda}{p_\lambda - h}}(\Lambda)}\right)^{\frac{p-2}{p-h}} \overset{B.3}{\leqslant} 2^{\frac{p-2}{p-h}-1} \left(1 + \|v\|^{\frac{s(p-2)}{p-h}}_{L^{\frac{s p_\lambda}{p_\lambda - h}}(\Lambda)}\right) = 2^{\frac{p-2}{p-h}-1} \left(1 + \|v\|^{s_1}_{L^{s_2}(\Lambda)}\right),$$

where

$$s_1 := s \cdot \frac{(p-2)}{p-h} \leqslant \frac{p(p-h)}{p-2} \cdot \frac{p-2}{p-h} = p$$

by our condition on s, and

$$\begin{aligned}
s_2 &:= s \cdot \frac{p_\lambda}{p_\lambda - h} \leqslant \frac{(h-2)p^2}{(p-2)(d-p)} \cdot \frac{p_\lambda}{p_\lambda - h} \overset{A.1}{=} \frac{dp}{d-p} = p_0 \quad \text{and} \\
s_2 &\geqslant 1 \cdot \underbrace{\frac{p_\lambda}{p_\lambda - h}}_{\geqslant 1} \geqslant 1.
\end{aligned}$$

Therefore $V \hookrightarrow L^{s_2}(\Lambda)$ by Theorem F.1 (i) with constant $C_{s_2,V} > 0$. Then Lemma A.2 with $s_1 \leqslant p$ and Lemma B.3 give

$$\begin{aligned}
\left(1 + \|v\|^{s_1}_{L^{s_2}(\Lambda)}\right) &\overset{F.1\,(i)}{\leqslant} \left(1 + C_{s_2,V} \|v\|_V^{s_1}\right) \\
&\leqslant \left(1 + C_{s_2,V}\left(1 + \|v\|_V\right)^{s_1}\right) \overset{A.2}{\leqslant} \left(1 + C_{s_2,V}\left(1 + \|v\|_V\right)^p\right) \\
&\overset{B.3}{\leqslant} \left(1 + 2^{p-1} C_{s_2,V}\left(1 + \|v\|_V^p\right)\right) \\
&\leqslant \left(1 + 2^{p-1} C_{s_2,V}\right)\left(1 + \|v\|_V^p\right).
\end{aligned}$$

Altogether by Lemma 3.2.1 and conditions (QL2) and (QL3) we get

$$
\begin{aligned}
&2\,{}_{V^*}\langle A_{p,f_0}(u)-A_{p,f_0}(v),u-v\rangle_V+\|B(u)-B(v)\|_{L_2}^2+\int_Z\|f(u,z)-f(v,z)\|_H^2\,m(\mathrm{d}z)\\
\leqslant&-2C_1\|u-v\|_V^p+2C_1\|u-v\|_V^p+\left(1+2^{p-1}C_{s_2,V}\right)2^{\frac{p-2}{p-h}}\tilde{C}\|u-v\|_{L^2(\Lambda)}^2\left(1+\|v\|_V^p\right)\\
&+C_{QL}\left(1+\|v\|_V^p\right)\|u-v\|_{L^2(\Lambda)}^2+C_{QL}\left(1+\|v\|_V^p\right)\|u-v\|_{L^2(\Lambda)}^2\\
=&\left(\left(1+2^{p-1}C_{s_2,V}\right)2^{\frac{p-2}{p-h}}\tilde{C}+2C_{QL}\right)\left(1+\|v\|_V^p\right)\|u-v\|_{L^2(\Lambda)}^2\\
\leqslant&\varrho(v)\|u-v\|_H^2\leqslant\left(F_t+\varrho(v)\right)\|u-v\|_H^2
\end{aligned}
$$

for all $t\in[0,T]$, since F is non-negative and $C_0\geqslant\left(1+2^{p-1}C_{s_2,V}\right)2^{\frac{p-2}{p-h}}\tilde{C}+2C_{QL}$.
Claim: (A3) holds.
Let $v\in V$. Condition (QL1) with $f_0(0)=0$ gives

$$
\begin{aligned}
{}_{V^*}\langle f_0(v),v\rangle_V={}&{}_{V^*}\langle f_0(v)-f_0(0),v-0\rangle_V=\int_\Lambda\left(f_0(v)-f_0(0)\right)(v-0)\,\mathrm{d}\xi\\
&\overset{(QL1)}{\leqslant}C_{QL}\int_\Lambda|v|^h\,\mathrm{d}\xi=C_{QL}\|v\|_{L^h(\Lambda)}^h.
\end{aligned}
$$

As in the previous claim we set

$$\lambda:=\frac{p(h-2)}{h(p-2)},\quad p_0:=\frac{dp}{d-p}>2.$$

Then $\lambda\in(0,1)$ and let p_λ such that

$$\frac{1}{p_\lambda}=\frac{1-\lambda}{2}+\frac{\lambda}{p_0}.$$

By Hölder's inequality with $q=\frac{p_\lambda}{p_\lambda-h}$ and $q'=\frac{p_\lambda}{h}$ and Lemma F.7 we obtain for $C_{QL,\Lambda}=C_{QL}|\Lambda|^{\frac{p_\lambda-h}{p_\lambda}}$

$$
\begin{aligned}
C_{QL}\|v\|_{L^h(\Lambda)}^h&\overset{Hölder}{\leqslant}C_{QL}|\Lambda|^{\frac{p_\lambda-h}{p_\lambda}}\|v\|_{L^{p_\lambda}(\Lambda)}^h\\
&\overset{F.7}{\leqslant}C_{QL,\Lambda}\|v\|_{L^2(\Lambda)}^{h(1-\lambda)}\|v\|_{L^{p_0}(\Lambda)}^{\lambda h}\\
&\overset{F.1\,(i)}{\leqslant}C_{QL,\Lambda}C_{p_0,V}^{\lambda h}\|v\|_{L^2(\Lambda)}^{h(1-\lambda)}\|v\|_V^{\lambda h},
\end{aligned}
$$

where $C_{p_0,V}>0$ is the constant from the continuous embedding $V\hookrightarrow L^{p_0}(\Lambda)=L^{\frac{dp}{d-p}}(\Lambda)$, cf. Theorem F.1 (i). Let $C_1>0$ the constant from Lemma 3.2.1 and set

$$\bar{C}_1:=\frac{1}{2}\frac{p}{\lambda h}C_1^{\frac{\lambda h}{p}}$$

Young's inequality with $q = \frac{p}{\lambda h}$ and $q' = \frac{p}{p-\lambda h}$ yields to

$$\begin{aligned} C_{QL,\Lambda} C_{p_0,V}^{\lambda h} \|v\|_{L^2(\Lambda)}^{h(1-\lambda)} \|v\|_V^{\lambda h} &= \left[\bar{C}_1^{-1} C_{QL,\Lambda} C_{p_0,V}^{\lambda h} \|v\|_{L^2(\Lambda)}^{h(1-\lambda)}\right] \left[\bar{C}_1 \|v\|_V^{\lambda h}\right] \\ &\overset{Young}{\leqslant} \frac{p-\lambda h}{p} \left[\bar{C}_1^{-1} C_{QL,\Lambda} C_{p_0,V}^{\lambda h} \|v\|_{L^2(\Lambda)}^{h(1-\lambda)}\right]^{\frac{p}{p-\lambda h}} + \frac{\lambda h}{p} \left[\bar{C}_1 \|v\|_V^{\lambda h}\right]^{\frac{p}{\lambda h}} \\ &= \tilde{C} \|v\|_{L^2(\Lambda)}^2 + \frac{1}{2} C_1 \|v\|_V^p \leqslant \tilde{C}\left(1 + \|v\|_H^2\right) + \frac{1}{2} C_1 \|v\|_V^p . \end{aligned}$$

Hence by Remark 3.2.2 and Lemma 3.2.1 we conclude

$$\begin{aligned} &2\,{}_{V^*}\langle A_{p,f_0}(v), v\rangle_V + \|B(v)\|_{L_2}^2 + (2C_1 - C_1)\|v\|_V^p \leqslant \left(\tilde{C} + L_B\right)\left(1 + \|v\|_H^2\right) \\ \leqslant & F_t + K\|v\|_H^2 \end{aligned}$$

for all $t \in [0,T]$, since $F_t \equiv C_0$, $K = C_0$ and $C_0 \geqslant \tilde{C} + L_B$.
Claim: (A4) holds.
Let $u \in V$. The operator norm for $A_{p,f_0}(u) : V \to \mathbb{R}$ with $\alpha = p$ can be estimated with (3.2.4) and Lemma B.3

$$\begin{aligned} \|A_{p,f_0}(u)\|_{V^*}^{\frac{\alpha}{\alpha-1}} &= \left[\sup_{\substack{v\in V,\\ \|v\|_V=1}} \left|{}_{V^*}\langle A_{p,f_0}(u), v\rangle_V\right|\right]^{\frac{p}{p-1}} \\ &\overset{(3.2.4)}{\leqslant} \left[\left(C_{QL}\left(C_{p,V}^2 + C_{1,V}\right) + 1\right)\left(1 + \|u\|_V^{p-1}\right)\right]^{\frac{p}{p-1}} \\ &\overset{B.3}{\leqslant} \left(C_{QL}\left(C_{p,V}^2 + C_{1,V}\right) + 1\right)^{\frac{p}{p-1}} \cdot 2^{-(p-1)}\left(1 + \|u\|_V^p\right) \\ &\leqslant \left(F_t + K\|u\|_V^\alpha\right) \underbrace{\left(1 + \|u\|_H^\beta\right)}_{\geqslant 1} \end{aligned}$$

for all $t \in [0,T]$ with $K = C_0$ and $C_0 \geqslant 2^{-(p-1)}\left(C_{QL}\left(C_{p,V}^2 + C_{1,V}\right) + 1\right)^{\frac{p}{p-1}}$ and since F is non-negative.
Claim: (A1), (B1), (B2) and (B3) holds.
We refer to the proof of Example 3.2.3, since α, β, γ and ϱ, K, F and $r = p-1$ are identical to the situation there and all the estimates are independent of d and do not involve s or h.
Claim: (3.2.1) has a solution.
Theorem 2.2.1 gives us a solution $X = (X_t)_{t\in[0,T]}$. With 2.2.1 (i) we have

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X_t\|_H^{2\zeta}\right] < \infty.$$

Claim: If $\eta = 0$, then the solution is unique.
Suppose $\gamma = \eta = 0$. Then by 2.2.1 (ii) our solution is unique and we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^{2\zeta}\right] < \infty.$$

□

3.2.5 Remark. *(i) Suppose $\zeta = 1$. Then in the proof of Examples 3.2.3 and 3.2.4 we have $\beta = 0$. Therefore, by Remark 2.3.6 (ii), we can even choose $\eta = \gamma < \theta$ to get existence, i.e.*

$$0 \leqslant \eta < \begin{cases} 2^{3-p}, & \textit{if } d < p, \\ 2^{2-p}, & \textit{else.} \end{cases}$$

(ii) To gather uniqueness in Examples 3.2.3 and 3.2.4 we do not need to assume that $\eta = 0$. We only need that $0 \leqslant \eta < \Gamma$, where Γ is the constant from (2.2.1) (cf. page 17), i.e. we can assume

$$0 \leqslant \eta < \frac{2^{4-p}\zeta}{4\zeta\left(\left(3C_{BDG}^2 + 2\right)\zeta + 4^\zeta - 1\right) - 3 \cdot 4^\zeta}$$

in Example 3.2.3 and, in Example 3.2.4,

$$0 \leqslant \eta < \frac{2^{3-p}\zeta}{4\zeta\left(\left(3C_{BDG}^2 + 2\right)\zeta + 4^\zeta - 1\right) - 3 \cdot 4^\zeta}.$$

Appendix

A. Supplements

The next Lemma is a supplement to the proof of (A2) in Example 3.2.4.

A.1 Lemma. *Let* $2 < p < d$, $2 < h < p$,

$$p_0 := \frac{dp}{d-p}, \quad \lambda := \frac{p\,(h-2)}{h\,(p-2)}$$

and p_λ *such that*

$$\frac{1}{p_\lambda} = \frac{1-\lambda}{2} + \frac{\lambda}{p_0}.$$

Then

(i) $h < p_\lambda$.

(ii) $\frac{(h-2)p^2}{(p-2)(d-p)} \cdot \frac{p_\lambda}{p_\lambda - h} = p_0$.

Proof. (i): From the definition of p_λ we deduce

$$p_\lambda = \frac{2p_0}{(1-\lambda)\,p_0 + 2\lambda}.$$

Then we see that

$$h < p_\lambda = \frac{2p_0}{(1-\lambda)\,p_0 + 2\lambda} \quad \Leftrightarrow \quad h\,(1-\lambda)\,p_0 + 2\lambda h < 2p_0.$$

But by the definition of λ and since $h-2>0$ and $p<d$

$$\begin{aligned}
&\frac{h\,(p-2) - p\,(h-2)}{p-2} p_0 + 2\frac{p\,(h-2)}{p-2} = h\,(1-\lambda)\,p_0 + 2\lambda h < 2p_0 \\
\Leftrightarrow\quad &\frac{2hp - 2hp_0 + 2pp_0 - 4p}{p-2} < 2p_0 \quad \Leftrightarrow \quad hp - hp_0 - 2p < -2p_0 \\
\Leftrightarrow\quad &p\,(h-2) < p_0\,(h-2) \quad \Leftrightarrow \quad p < p_0 = \frac{dp}{d-p} \quad \Leftrightarrow \quad dp - p^2 < dp
\end{aligned}$$

and this is obviously true, so (i) holds.
(ii): First we calculate

$$\begin{aligned}
\frac{p_\lambda}{p_\lambda - h} &= 2p_0\,[2p_0 - (1-\lambda)\,p_0 h - 2\lambda h]^{-1} \\
&= 2p_0\,(p-2)\,[2p_0\,(p-2) - hp_0\,(p-2) + p_0 p\,(h-2) - 2p\,(h-2)]^{-1} \\
&= p_0\,(p-2)\,[(h-2)\,(p_0-p)]^{-1}.
\end{aligned}$$

Then

$$\frac{p_\lambda}{p_\lambda - h} \cdot \frac{(h-2)\,p^2}{(p-2)\,(d-p)} = \frac{p_0 p^2}{(p_0 - p)\,(d-p)} = p_0$$

because

$$p^2 = p^2 + dp - dp = dp - p\,(d-p) = \left(\frac{dp}{d-p} - p\right)(d-p) = (p_0 - p)\,(d-p)\,.$$

□

A.2 Lemma. *Let* $1 \leqslant a < \infty$ *and* $0 \leqslant p \leqslant q < \infty$*. Then*

$$a^p \leqslant a^q.$$

Proof.

$$a^p = \exp\left(\ln\left(a^p\right)\right) = \exp\left(p \ln a\right) \overset{\ln a \geqslant 0}{\leqslant} \exp\left(q \ln a\right) = a^q.$$

□

A.3 Lemma. *Let* $d \in \mathbb{N}$*,* $\Lambda \subset \mathbb{R}^d$ *an open, bounded domain. Consider the Gelfand triple*

$$V := H_0^{1,2}(\Lambda) \subset H := L^2(\Lambda) \subset \left(H_0^{1,2}(\Lambda)\right)^* := V^*.$$

Let $C_0^\infty(\Lambda)$ *be the set of all infinitely differentiable real-valued functions with compact support in* Λ*. Then the Laplace operator* $\Delta\colon C_0^\infty(\Lambda) \to C_0^\infty(\Lambda)$ *extends uniquely to an operator* $A\colon V \to V^*$ *and we have*

$$\left|{}_{V^*}\langle \Delta u, v\rangle_V\right| \leqslant \|u\|_V \, \|v\|_V\,.$$

Proof. Cf. [PR07, Example 4.1.7]. Since $C_0^\infty(\Lambda) \subset L^2(\Lambda) \subset \left(H_0^{1,2}(\Lambda)\right)^*$, we have $\Delta\colon C_0^\infty(\Lambda) \to V^*$. Further let us note that $C_0^\infty(\Lambda)$ is dense in $H_0^{1,2}(\Lambda)$. By integrating by parts and Hölder's inequality, we have for $u, v \in C_0^\infty(\Lambda)$

$$\begin{aligned}\left|{}_{V^*}\langle \Delta u, v\rangle_V\right| = \left|\langle \Delta u, v\rangle_H\right| &= \left|-\int_\Lambda \langle \nabla u(\xi), \nabla v(\xi)\rangle \,\mathrm{d}\xi\right| \\ \leqslant \left(\int_\Lambda |\nabla u(\xi)|^2 \,\mathrm{d}\xi\right)^{\frac{1}{2}} &\left(\int_\Lambda |\nabla v(\xi)|^2 \,\mathrm{d}\xi\right)^{\frac{1}{2}} = \|u\|_V\,\|v\|_V\,.\end{aligned}$$

Hence we have

$$\|\Delta u\|_{V^*} \leqslant \|u\|_V$$

and therefore Δ extends uniquely to an operator $A\colon V \to V^*$ with $A = \Delta$.

□

B. Inequalities

The most important inequalities that we used throughout this thesis are collected in this section.

B.1 Lemma (Young's inequality)**.** *Let* $a, b \geqslant 0$*,* $1 < p < \infty$*,* $p' = \frac{p}{p-1}$*. Then*

$$ab \leqslant \frac{1}{p} a^p + \frac{1}{p'} b^{p'}.$$

Proof. See [Alt06, (1-11), p. 52]. □

B.2 Lemma (Generalized Hölder's inequality)**.** *Let* $(X, \mathcal{X}, \mu)$ *a measurable space,* $m \in \mathbb{N}$ *and* $p_i, q \in [1, \infty]$ *for* $i \in \{1, \ldots, m\}$ *such that*

$$\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{q},$$

with $\frac{1}{r} = 0$ *if* $r = \infty$*. Further let* $u_i \in L^{p_i}(X, \mu; \mathbb{R})$ *for all* $i \in \{1, \ldots, m\}$*. Then we have* $u_1 \cdots u_m \in L^q(X, \mu; \mathbb{R})$ *and*

$$\left\| \prod_{i=1}^{m} u_i \right\|_{L^q(X,\mu;\mathbb{R})} \leqslant \prod_{i=1}^{m} \|u_i\|_{L^{p_i}(X,\mu;\mathbb{R})}.$$

Proof. See [Alt06, Lemma 1.16]. □

B.3 Lemma. *Let* $a, b \in \mathbb{R}$ *and* $1 \leqslant p < \infty$*. Then*

$$(a+b)^p \leqslant 2^{p-1}(a^p + b^p).$$

Proof. Since $p \geqslant 1$, the mapping $x \mapsto x^p$ is convex. With Jensen's inequality we get

$$(a+b)^p = 2^p \left(\frac{1}{2} a + \frac{1}{2} b \right)^p \leqslant 2^p \left(\frac{1}{2} a^p + \frac{1}{2} b^p \right) = 2^{p-1}(a^p + b^p).$$

□

B.4 Lemma (Bihari's inequality)**.** *Let* $g\colon [0, \infty[\to [0, \infty[$ *be a non-decreasing, continuous function with* $g((0, \infty)) \subset (0, \infty)$*. Let* $A \geqslant 0$ *and suppose* $f, h\colon [0, \infty[\to [0, \infty[$ *are measurable functions with* $h \in L^1_{loc}([0, \infty[)$ *and such that for all* $t \geqslant 0$ *we have*

$$f(t) \leqslant A + \int_0^t h(s)\, g(f(s))\, \mathrm{d}s. \tag{B.1}$$

Let $x_0 \in (0, \infty)$ *fixed such that* $G(x) := \int_{x_0}^{x} \frac{\mathrm{d}s}{g(s)} < \infty$ *for all* $x \in (0, \infty)$*. Furthermore let* $T_0 \in (0, \infty)$ *such that* $G(A) + \int_0^{T_0} h(s)\, \mathrm{d}s$ *belongs to the domain of the inverse function of* G*, namely* $G^{-1}\colon G((0,1)) \to (0, \infty)$*. Then for all* $0 \leqslant t \leqslant T_0$ *we have*

$$f(t) \leqslant G^{-1}\left(G(A) + \int_0^t h(s)\, \mathrm{d}s \right). \tag{B.2}$$

Proof. See [LR13, Lemma 2.1] and [Bih56]. □

From Bihari's inequality we can deduce the well known Gronwall's inequality even for non-continuous, but measurable functions.

B.5 Lemma (Gronwall's inequality)**.** *Let $0 < T < \infty$ and let $f\colon [0,T] \to \mathbb{R}$ measurable and non-negative. Let $A \geqslant 0$, $b \in (0,\infty)$ such that*

$$f(t) \leqslant A + \int_0^t bf(s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{B.3}$$

Then for all $t \in [0,T]$

$$f(t) \leqslant Ae^{bt}.$$

Proof. Consider $g(x) := bx$, $h \equiv 1$. Then g is continuous and, since $b > 0$, g is non-decreasing and obeys $g((0,\infty)) \subset (0,\infty)$. In the situation of Lemma B.4 we set $x_0 = 1$ and see that

$$G(x) = \int_1^x \frac{1}{g(s)}\,\mathrm{d}s = \frac{1}{b}\left[\ln(\cdot)\right]_1^x = \frac{\ln x}{b} < \infty$$

for all $x \in (0,\infty)$. The inverse function of G is given by $G^{-1}(y) = e^{by}$, because

$$G^{-1}(G(x)) = e^{b\frac{\ln x}{b}} = x \quad \text{for all } x \in (0,\infty).$$

Therefore we can choose $T_0 := T$, see also [LR10, Remark 2.1]. In this setup, (B.3) implies (B.1):

$$f(t) \leqslant A + \int_0^t bf(s)\,\mathrm{d}s = A + \int_0^t h(s)\,g(f(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T].$$

Then by Bihari's inequality we get for all $0 \leqslant t \leqslant T$

$$\begin{aligned} f(t) &\overset{\text{(B.2)}}{\leqslant} G^{-1}\left(G(A) + \int_0^t h(s)\,\mathrm{d}s\right) = e^{b\left(\frac{\ln A}{b}+t\right)} \\ &= Ae^{bt}. \end{aligned}$$

□

C. Inequalities on Hilbert spaces

Let $(H, \|\cdot\|_H)$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and standard norm $\|\cdot\|_H = \sqrt{\langle\cdot,\cdot\rangle_H}$.

C.1 Lemma. *Let $p \in [2,\infty)$. There exists a constant $C = C(p) > 0$ such that for all $x, y \in H$ we have*

$$\left|\|x+h\|_H^p - \|x\|_H^p - p\|x\|_H^{p-2}\langle x,h\rangle_H\right| \leqslant C\left(\|x\|_H^p + \|h\|_H^p\right).$$

If $p = 2$ then we have

$$\left|\|x+h\|_H^2 - \|x\|_H^2 - 2\langle x,h\rangle_H\right| = \|h\|_H^2.$$

C.2 Remark. *By [MR13, Lemma 2.2] Lemma C.1 holds true for $1 \leqslant p \leqslant 2$. More precisely, there exists a constant $C = C(p) > 0$ such that for any $x, y \in H$ we have*

$$\|x+y\|_H^p - \|x\|_H^p - p\,\|x\|_H^p \langle x, y\rangle_H \leqslant C\,\|y\|_H^p .$$

To prove Lemma C.1 we need the following Lemma.

C.3 Lemma. *Let $p < \infty$. For any $x, y \in \mathbb{R}$ the following inequalities hold:*

(i) $|x+y|^p \leqslant 2^{p-1}\,|x^p + y^p|$, *if* $p \geqslant 1$.

(ii) $(x+y)^p - x^p \leqslant 2^{p-2}\left(p\,yx^{p-1} + y^p\right) \quad \Big(\leqslant 2^{p-1}\left(p\,yx^{p-1} + y^p\right)\Big)$, *if* $p \geqslant 2$.

Proof. (i) For $p = 1$ there is nothing to show. For $p > 1$ we use the generalized Hölder inequality to see that

$$|x+y|^p = |1\cdot x + 1\cdot y|^p \leqslant \left(|1+1|^{\frac{p-1}{p}}\,|x^p+y^p|^{\frac{1}{p}}\right)^p = 2^{p-1}\,|x^p+y^p| .$$

(ii) We use the fundamental theorem of calculus and (i) to obtain

$$\begin{aligned}(x+y)^p - x^p &= \int_x^{x+y} \frac{\mathrm{d}}{\mathrm{d}t}\left(t^p\right)\,\mathrm{d}t = p\int_x^{x+y} t^{p-1}\,\mathrm{d}t \\ &= p\int_0^y (x+t)^{p-1}\,\mathrm{d}t \overset{(i)}{\leqslant} p2^{p-2}\int_0^y \left(x^{p-1}+t^{p-1}\right)\,\mathrm{d}t \\ &= 2^{p-2}\left(pyx^{p-1} + \int_0^y pt^{p-1}\,\mathrm{d}t\right) = 2^{p-2}\left(p\,yx^{p-1}+y^p\right).\end{aligned}$$

□

Proof of Lemma C.1. For given and fixed $p \in [2,\infty)$ and $x, h \in H$ we set $g(t) = \|x+th\|_H^p$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = p\,\|x+th\|_H^{p-2}\left(t\|h\|_H^2 + \langle x, h\rangle_H\right)$$

and

$$\Big|\|x+h\|_H^p - \|x\|_H^p - p\|x\|_H^{p-2}\langle x, h\rangle_H\Big| = \left|g(1) - g(0) - g'(0)\right| \leqslant |g(1)-g(0)| + \left|g'(0)\right| .$$

First step: We first apply the Cauchy-Schwarz-inequality to $|g'(0)|$ and then Young's inequality to obtain

$$\begin{aligned}\left|g'(0)\right| &= p\|x\|_H^{p-2}\left|\langle x, h\rangle_H\right| \overset{C.-S.}{\leqslant} p\|x\|_H^{p-1}\|h\|_H \\ &\overset{Young}{\leqslant} p\left(\frac{p-1}{p}\|x\|_H^p + \frac{1}{p}\|x\|_H^p\right) \leqslant C_1\left(\|x\|_H^p + \|h\|_H^p\right),\end{aligned}$$

where $C_1 = C_1(p) = p - 1$.

Second step: According to the mean value theorem there exists $t_0 \in [0,1]$ such that $g(1) - g(0) = g'(t_0)$. Using the triangle inequality we have

$$
\begin{aligned}
\frac{1}{p}\left|g'(t_0)\right| &= \|x + t_0 h\|_H^{p-2}\left|t_0\|h\|_H^2 + \langle x, h\rangle_H\right| \leqslant \left(\|x\|_H + t_0\|h\|_H\right)^{p-2}\left(t_0\|h\|_H^2 + |\langle x, h\rangle_H|\right) \\
&\leqslant \left(\|x\|_H + \sup_{t\in[0,1]} t\,\|h\|_H\right)^{p-2}\left(\sup_{t\in[0,1]} t\,\|h\|_H^2 + |\langle x, h\rangle_H|\right) \\
&= \left(\|x\|_H + \|h\|_H\right)^{p-2}\|h\|_H^2 + \left(\|x\|_H + \|h\|_H\right)^{p-2}|\langle x, h\rangle_H|. \qquad \text{(C.1)}
\end{aligned}
$$

Now we apply Young's inequality to the first summand of (C.1) and then Lemma C.3 and obtain

$$
\begin{aligned}
\left(\|x\|_H + \|h\|_H\right)^{p-2}\|h\|_H^2 &\leqslant \frac{p-2}{p}\left(\|x\|_H + \|h\|_H\right)^p + \frac{2}{p}\|h\|_H^p \\
&\overset{C.3}{\leqslant} 2^{p-1}\frac{p-2}{p}\left(\|x\|_H^p + \|h\|_H^p\right) + \frac{2}{p}\|h\|_H^p \\
&\leqslant C_2\left(\|x\|_H^p + \|h\|_H^p\right)
\end{aligned}
$$

with $C_2 = C_2(p) = 2^{p-1}\frac{p-1}{p}$ since $p - 1 \geqslant 1$ implies $2^{p-1}(p-1) \geqslant 2$ and $2^{p-1}(p-1) \geqslant 2^{p-1}(p-2)$.

Again, the Cauchy-Schwarz inequality, Young's inequality (used two times) and Lemma C.3 used on the second summand of (C.1) imply

$$
\begin{aligned}
\left(\|x\|_H + \|h\|_H\right)^{p-2}|\langle x, h\rangle_H| &\overset{C.-S.}{\leqslant} \left(\|x\|_H + \|h\|_H\right)^{p-2}\|x\|_H\|h\|_H \\
&\leqslant \frac{p-2}{p}\left(\|x\|_H + \|h\|_H\right)^p + \frac{2}{p}\|x\|_H\|h\|_H \\
&\overset{C.3}{\leqslant} 2^{p-1}\frac{p-2}{p}\left(\|x\|_H^p + \|h\|_H^p\right) + \frac{2}{p}\|x\|_H^{\frac{p}{2}}\|h\|_H^{\frac{p}{2}} \\
&\leqslant 2^{p-1}\frac{p-2}{p}\left(\|x\|_H^p + \|h\|_H^p\right) + \frac{2}{p}\left(\frac{1}{2}\|x\|_H^p + \frac{1}{2}\|h\|_H^p\right) \\
&= C_3\left(\|x\|_H^p + \|h\|_H^p\right),
\end{aligned}
$$

where $C_3 = C_3(p) = \frac{1}{p}\left(2^{p-1}(p-2) + 1\right)$.

Third step: We combine the results from step one and two and finally have

$$
|g(1) - g(0)| + \left|g'(0)\right| \leqslant \left(C_1 + pC_2 + pC_3\right)\left(\|x\|_H^p + \|h\|_H^p\right).
$$

Setting $C = (C_1 + pC_2 + pC_3) = p + 2^{p-1}(2p-3)$, we finish the proof of the first statement.

Fourth step: The second statement follows immediately by

$$
\|x + h\|_H^2 = \|x\|_H^2 + \|h\|_H^2 + 2\langle x, h\rangle_H.
$$

□

C.4 Lemma. *For any $p \geqslant 0$ and $x, y \in H$ we have*

$$
\left\langle \|x\|_H^p\, x - \|y\|_H^p\, y, x - y\right\rangle_H \geqslant 2^{-p}\|x - y\|_H^{p+2}.
$$

Proof. See [Liu09, Lemma 3.1]. □

D. Tools on processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)_{t\in[0,T]}$, $0 < T < \infty$, be a filtration on $\mathcal{F}$. Let $(H, \|\cdot\|_H)$ be a separable Hilbert space and $\mathcal{M}_T^2(H)$ the space of all square integrable martingales on H with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ up to time T.

D.1 Proposition. *Let $M \in \mathcal{M}_T^2(H)$. Then there exists a unique predictable process $\langle M\rangle$ of bounded variation such that*

$$\|M(t)\|_H^2 - \langle M\rangle_t, \quad t \geqslant 0,$$

is a martingale.

Proof. See [PZ07, Remark 3.46]. □

D.2 Proposition. *Let M a càdlàg, local $(\mathcal{F}_t)$-martingale in H. Let τ_n a sequence of partitions $\left\{0 \leqslant t_1^n < \cdots < t_{m_n}^n\right\}$ such that $\lim_{n\to\infty} t_{m_n}^n \to \infty$ and the for the mesh of τ_n we have $\lim_{n\to\infty} \sup_{1\leqslant k\leqslant m_n-1} \left|t_k^n - t_{k+1}^n\right| = 0$. Then*

$$\sum_{k=1}^{m_n-1} \left\|M_{t_k^n\wedge t} - M_{t_{k+1}^n\wedge t}\right\|_H^2$$

converges in $L^1(\Omega, P)$.

Proof. See [Mé82, Theorem 18.6]. □

D.3 Definition. The limiting process in the previous Proposition is denoted by

$$[M]_t$$

for $t \in [0, T]$ and called the *square bracket of* M.

The next theorem will show that hitting times of càdlàg processes are stopping times.

D.4 Theorem. *Let X be an $\mathcal{F}_t$-adapted right-continuous process with values in H. Then*

$$\tau_R = \inf\{t \geqslant 0 \,|\, \|X(t)\| > R\}, \quad R > 0,$$

is a stopping time.

Proof. See [Kal97, Theorem 6.7]. □

D.5 Theorem (Burkholder-Davis-Gundy inequality)**.** *Let $(M_t)_{t\geqslant 0}$ be a real-valued, càdlàg local martingale on a probability space $(\Omega, \mathcal{F}, P)$ with respect to a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ with $M_0 = 0$ and let $p \geqslant 1$.*

(i) There exists a constant $C = C(p) > 0$ such that for every stopping time $\tau \leqslant T$ we have

$$\mathbb{E}\left[\sup_{t\in[0,\tau]} |M_t|^p\right] \leqslant C\,\mathbb{E}\left[[M]_\tau^{\frac{p}{2}}\right].$$

(ii) If $\mathbb{E}\left[[M]_T^{\frac{1}{2}}\right] < \infty$*, then* $(M_t)_{t\in[0,T]}$ *is a martingale.*

Proof. **Part (i)**: Apply [Kal97, Theorem 23.12] to the stopped process $(M_{t\wedge\tau})_{t\in[0,T]}$. (Also see [KS91, Theorem 3.28] for the continuous case.)
Part (ii): Cf. [Kal97, Corollary 15.9]. This proof is quoted from a newer, not yet published version of [PR07, Proposition D.0.1]: Let $\tau_n\colon \Omega \to [0,T]$ be a sequence of stopping times such that $(M_{t\wedge\tau_n})_{t\in[0,T]}$ is a martingale and $\lim_{n\to\infty}\tau_n = T$. Then for all $t\in[0,T]$

$$\lim_{n\to\infty} M_{t\wedge\tau_n} = M_t \quad P\text{-a.s.}$$

By (i) we have

$$\sup_{n\in\mathbb{N}} |M_{t\wedge\tau_n}| \leqslant \sup_{s\in[0,T]} |M_s| \in L^1(\Omega;\mathbb{R}).$$

Lebesgue's dominated convergence theorem hence gives us for all $t\in[0,T]$

$$\lim_{n\to\infty} M_{t\wedge\tau_n} = M_t \quad \text{in } L^1(\Omega;\mathbb{R})$$

and so (ii) follows.

□

E. Miscellaneous tools

E.1 Theorem (Banach-Alaoglu)**.** *Let* X *be a Banach space. Then the closed unit ball*

$$X^* \supset \overline{B_1(0)} = \{f \in X^* \mid \|f\|_{X^*} \leqslant 1\}$$

is weakly-star compact.

Proof. See [Bre10, Theorem 3.16]. □

E.2 Lemma. *Let* $(\Omega,\mathcal{F},P)$ *be a probability space and* $0 < T < \infty$*. Let* $f\colon \Omega\times[0,T]\to\mathbb{R}$ *right lower semicontinuous. Then*

$$\operatorname*{ess\,sup}_{t\in[0,T]} f(t) = \sup_{t\in[0,T]} f(t).$$

Proof. By the definition of the essential supremum, we always have $\operatorname{ess\,sup} f \leqslant \sup f$. Therefore we suppose that

$$\alpha := \operatorname*{ess\,sup}_{t\in[0,T]} f(t) < \sup_{t\in[0,T]} f(t).$$

Then there exists $\delta > 0$ such that

$$\alpha - \delta < \sup_{t\in[0,T]} f(t)$$

and for all $n \in \mathbb{N}$ we can find $t_n \in [0,T]$ with

$$\alpha - \delta - \frac{1}{n} < f(t_n).$$

Since f is right lower semicontinuous, for all $n \in \mathbb{N}$ there exists an $\varepsilon_n > 0$ such that

$$\alpha - \delta - \frac{1}{n} < f(t) \quad \text{for all } t \in [t_n, t_n + \varepsilon_n).$$

This set is not a Lebesgue-zero-set, hence for all $n \in \mathbb{N}$ we can find $s_n \in [t_n, t_n + \varepsilon_n)$ with

$$f(s_n) \leqslant \operatorname*{ess\,sup}_{t \in [0,T]} f(t) \quad \Big(= \alpha\Big).$$

Now

$$\alpha - \delta - \frac{1}{n} < f(s_n) \leqslant \alpha.$$

Letting $n \to \infty$ we see the contradiction $\alpha - \delta < \alpha$. Hence the assertion follows.

□

F. Some important embeddings and interpolations

The next theorem summarizes the most important Sobolev embeddings.

F.1 Theorem. *Let $d \in \mathbb{N}$, $\Lambda \subset \mathbb{R}^d$ open and bounded. Let $m, n \in \mathbb{N}_0$ and $1 \leqslant p, q < \infty$ and set $H_0^{0,p}(\Lambda) := L^p(\Lambda)$.*

(i) If

$$m - \frac{d}{p} \geqslant n - \frac{d}{q} \quad \textit{and} \quad m \geqslant n,$$

then there exists a constant $C = C(m,n,p,q,d,|\Lambda|) > 0$ such that for all $u \in H_0^{m,p}(\Lambda)$ we have

$$\|u\|_{H_0^{n,q}(\Lambda)} \leqslant C \|u\|_{H_0^{m,p}(\Lambda)}.$$

In other words, $H_0^{m,p}(\Lambda) \hookrightarrow H_0^{n,q}(\Lambda)$ is a continuous embedding.

(ii) If

$$m - \frac{d}{p} > n - \frac{d}{q} \quad \textit{and} \quad m > n,$$

then the embedding $H_0^{m,p}(\Lambda) \hookrightarrow H_0^{n,p}(\Lambda)$ is continuous and compact, i.e. $H_0^{m,p}(\Lambda) \Subset H_0^{n,p}(\Lambda)$.

Proof. See [Alt06, 8.9 Einbettungssatz in Sobolev-Räumen, p. 328]. □

F.2 Remark. *In the situation of Theorem F.1, all the embeddings hold for the Sobolev spaces $H^{m,p}(\Lambda)$ if $\Lambda \subset \mathbb{R}^d$ is open, bounded and has smooth boundary.*

F.3 Proposition. *Let $d \in \mathbb{N}$, $\Lambda \subset \mathbb{R}^d$ open and bounded and $(1 \leqslant)\ d < p < \infty$. Then there exists a constant $C = C(d, p, |\Lambda|) > 0$ such that for all $u \in H_0^{1,p}(\Lambda)$*

$$\|u\|_{L^\infty(\Lambda)} \leqslant C \|\nabla u\|_{L^p(\Lambda)}.$$

Proof. See [Alt06, 8.10 Satz, p. 330]. □

An important special case is Poincaré's inequality:

F.4 Corollary (Poincaré)**.** *Let $d \in \mathbb{N}$, $\Lambda \subset \mathbb{R}^d$ open and bounded and $1 \leqslant p < \infty$. Then there exists a constant $C = C(p, d, |\Lambda|) > 0$ such that for all $u \in H_0^{1,p}(\Lambda; \mathbb{R})$*

$$\int_\Lambda |u|^p \,\mathrm{d}\xi \leqslant C \int_\Lambda |\nabla u|^p \,\mathrm{d}\xi.$$

Proof. Apply Theorem F.1 with $m = 1$, $n = 0$, $p = q$. □

F.5 Lemma. *Let $d \in \mathbb{N}$, $\Lambda \subset \mathbb{R}^d$ open and bounded and let $1 \leqslant p < \infty$. Set $L^p := L^p(\Lambda) := L^p(\Lambda, \mathbb{R})$ and $H_0^{m,p}(\Lambda) := H_0^{m,p}(\Lambda, \mathbb{R})$ for $m \in \mathbb{N}$.*

(i) If $d = 2$, then for all $u \in H_0^{1,2}$

$$\|u\|_{L^4}^4 \leqslant 4 \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$

(ii) If $d = 3$, then for all $u \in H_0^{1,2}$

$$\|u\|_{L^4}^4 \leqslant 8 \|u\|_{L^2} \|\nabla u\|_{L^2}^3.$$

Proof. See [MS02, Lemma 2.1] and succeeding remark. □

F.6 Lemma. *Let $\Lambda \subset \mathbb{R}^d$ be an open, bounded domain with $d \in \mathbb{N}$ and $d \geqslant 3$. Let $g \in L^d(\Lambda) + L^\infty(\Lambda)$ and $\varepsilon > 0$. Then there exists a constant $C = C(\varepsilon, d, |\Lambda|) > 0$ such that for all $u, v \in H_0^{1,2}(\Lambda)$*

$$\int_\Lambda |g|\,|u|\,|\nabla v| \,\mathrm{d}\xi \leqslant \left(\varepsilon \|u\|^2_{H_0^{1,2}(\Lambda)} + C \|u\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}} \left(\varepsilon \|v\|^2_{H_0^{1,2}(\Lambda)} + C \|v\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}}.$$

In particular, for $\varepsilon = 1$ we have $C = C(d, |\Lambda|)$ and

$$\int_\Lambda |g|\,|u|\,|\nabla v| \,\mathrm{d}\xi \leqslant \left(\|u\|^2_{H_0^{1,2}(\Lambda)} + C \|u\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}} \left(\|v\|^2_{H_0^{1,2}(\Lambda)} + C \|v\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}}.$$

Proof. This proof is taken from [LR14]. Define

$$C_0 := \frac{1}{\varepsilon} \inf \left\{ R \in (0, \infty) \,\middle|\, \left\| \mathbb{I}_{\{|g|^2 > R\}} g \right\|_{L^d(\Lambda)} \leqslant \frac{\varepsilon}{C_1} \right\},$$

where $C_1 = C_1(d, |\Lambda|) > 0$ is the constant from Theorem F.1 (i) for $n = 0$, $q = \frac{2d}{d-2}$, $m = 1$, $p = 2$. Since $g \in L^d(\Lambda) + L^\infty(\Lambda)$, we have $C_0 < \infty$. By Hölder's inequality with $q = 2$ we get

$$\int_\Lambda |g|\,|u|\,|\nabla v|\ \mathrm{d}\xi \leqslant \left(\int_\Lambda |g|^2 |u|^2\ \mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_\Lambda |\nabla v|^2\ \mathrm{d}\xi\right)^{\frac{1}{2}}.$$

Again, Hölder's inequality with $q = \frac{d}{2}$, $q' = \frac{d}{d-2}$

$$\begin{aligned}
\left(\int_\Lambda |g|^2 |u|^2\ \mathrm{d}\xi\right)^{\frac{1}{2}} &\leqslant \left(\int_\Lambda \mathbb{I}_{\{|g|^2 > C_0\varepsilon\}} |g|^2 |u|^2\ \mathrm{d}\xi + C_0\varepsilon \|u\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}} \\
&\overset{Hölder}{\leqslant} \left(\left(\int_\Lambda \mathbb{I}_{\{|g|^2 > C_0\varepsilon\}} |g|^{2\frac{d}{2}}\ \mathrm{d}\xi\right)^{\frac{2}{d}} \left(\int_\Lambda |u|^{2\frac{d}{d-2}}\ \mathrm{d}\xi\right)^{\frac{d-2}{d}} + C_0\varepsilon \|u\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}} \\
&= \left(\left\|\mathbb{I}_{\{|g|^2 > C_0\varepsilon\}} g\right\|^2_{L^d(\Lambda)} \|u\|^2_{L^{\frac{2d}{d-2}}(\Lambda)} + C_0\varepsilon \|u\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}} \\
&\leqslant \left(\frac{\varepsilon^2}{C_1^2} \cdot C_1^2 \int_\Lambda |\nabla u|^2\ \mathrm{d}\xi + C_0\varepsilon \|u\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}}
\end{aligned}$$

by the definition of C_0 and C_1. Now, since $\int_\Lambda |\nabla v|^2\ \mathrm{d}\xi \leqslant \int_\Lambda |\nabla v|^2\ \mathrm{d}\xi + \frac{1}{\varepsilon} C_0 \|v\|^2_{L^2(\Lambda)}$, we have

$$\begin{aligned}
\int_\Lambda |g|\,|u|\,|\nabla v|\ \mathrm{d}\xi &\leqslant \left(\varepsilon^2 \int_\Lambda |\nabla u|^2\ \mathrm{d}\xi + C_0\varepsilon \|u\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}} \left(\int_\Lambda |\nabla v|^2\ \mathrm{d}\xi + \frac{1}{\varepsilon} C_0 \|v\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}} \\
&= \varepsilon^{\frac{1}{2}} \left(\varepsilon \int_\Lambda |\nabla u|^2\ \mathrm{d}\xi + C_0 \|u\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}} \left(\varepsilon \int_\Lambda |\nabla v|^2\ \mathrm{d}\xi + C_0 \|v\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}}
\end{aligned}$$

which implies the assertion for $C = C_0(\varepsilon, d, |\Lambda|)$.

□

The following Lemma is also known as the log-convexity of L^p-norms.

F.7 Lemma (Interpolation of L^p-norms)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. For $0 < p < \infty$ we denote $L^p(X) := L^p(X, \mu; \mathbb{R})$. Let $0 < p_0 < p_1 < \infty$ and $u \in L^{p_0}(X) \cap L^{p_1}(X)$. Then*

$$u \in L^p(X) \quad \textit{for all } p_0 \leqslant p \leqslant p_1.$$

Moreover, we have for all $0 \leqslant \lambda \leqslant 1$

$$\|u\|_{L^{p_\lambda}(X)} \leqslant \|u\|^{1-\lambda}_{L^{p_0}(X)} \|u\|^{\lambda}_{L^{p_1}(X)},$$

where $\frac{1}{p_\lambda} := \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}$.

Proof. Let $0 \leqslant \lambda \leqslant 1$. We set

$$q := \frac{p_0}{(1-\lambda)\,p_\lambda} = \left(\frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}\right)\frac{p_0}{1-\lambda} = 1 + \frac{\lambda p_0}{(1-\lambda)\,p_1} = \frac{(1-\lambda)\,p_1 + \lambda p_0}{(1-\lambda)\,p_1}.$$

Then for the dual of q we have

$$q' = \frac{(1-\lambda)\,p_1 + \lambda p_0}{\lambda p_0} = 1 + \frac{(1-\lambda)\,p_1}{\lambda p_0} = \frac{p_1}{\lambda}\left(\frac{\lambda}{p_1} + \frac{1-\lambda}{p_0}\right) = \frac{p_1}{\lambda p_\lambda}.$$

Now by Hölder's inequality

$$\begin{aligned}\|u\|_{L^{p_\lambda}(X)} &= \left(\int_X |u|^{(1-\lambda)p_\lambda}\,|u|^{\lambda p_\lambda}\;\mathrm{d}\mu\right)^{\frac{1}{p_\lambda}} \leqslant \left(\int_X |u|^{(1-\lambda)p_\lambda q}\;\mathrm{d}\mu\right)^{\frac{1}{q}\frac{1}{p_\lambda}} \left(\int_X |u|^{\lambda p_\lambda q'}\;\mathrm{d}\mu\right)^{\frac{1}{q'}\frac{1}{p_\lambda}} \\ &= \left(\int_X |u|^{p_0}\;\mathrm{d}\mu\right)^{\frac{(1-\lambda)p_\lambda}{p_0}\frac{1}{p_\lambda}} \left(\int_X |u|^{p_1}\;\mathrm{d}\mu\right)^{\frac{\lambda p_\lambda}{p_1}\frac{1}{p_\lambda}} = \|u\|_{L^{p_0}(X)}^{1-\lambda}\,\|u\|_{L^{p_1}(X)}^{\lambda}\,,\end{aligned}$$

and the second assertion follows. Since λ is arbitrary, the first assertion follows. □

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